

MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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APRIL 13, 1928.

No. 6

Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

The President read the following resolution of the Council, which was approved by the Meeting :—

The President, Council, and Fellows of the Royal Astronomical Society have heard with the greatest regret of the death of Mr. E. Walter Maunder, for fifty-three years a Fellow of the Society, Secretary from 1892 to 1897, and Vice-President in 1897. They wish to place on record their sense of the value of his work on behalf of the Society, and of his contributions to Astronomy, including that of founding the British Astronomical Association ; and at the same time to express their deep sympathy with Mrs. Maunder and with his family in their bereavement.

The election by the Council of the following Fellows was duly confirmed :—

Miss Ida Barney, A.B., Ph.D., Yale University Observatory, New Haven, Conn., U.S.A. ;

The Rev. Samuel Martin Johnstone, B.A., St. John's Rectory, Parramatta, N.S.W., Australia ;

William Morton Page, C.B.E., M.A., H.M. Inspector, Board of Education, Norway House, Lostwithiel, Cornwall ;

David B. Pickering, 171 South Burnett Street, East Orange, New Jersey, U.S.A. ; and

Ernest Tillotson, B.Sc., 23 Roseville Road, Leeds.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

James Currie, M.A., Martnaham, Cardonald, Glasgow, S.W. 2
(proposed by T. L. MacDonald) ;

Wilfred Denne, Dingle Cottage, Sheepcote Lane, Watford, Herts (proposed by J. S. Fletcher);
 Charles W. Elmer, 15 Park Row Building, New York City, U.S.A. (proposed by Anne S. Young); and
 Harold Henry Thorne, M.A., B.Sc., The University of Sydney, Sydney, N.S.W.; and 68 Lambton Road, London, S.W. 20 (proposed by S. Chapman).

The following were proposed by the Council as Associates of the Society:—

Dr. Edwin Hubble, Mount Wilson Observatory, California, U.S.A.;
 General Gustave Ferrié, Directeur des Services de la Télégraphie Militaire, Paris; and
 Captain Leon Herrero, The Observatory, San Fernando.

Ninety-one presents were announced as having been received since the last meeting, including, among others:—

Carte photographique du Ciel: Bordeaux Observatory, zone $+12^{\circ}$ (9 charts); Paris Observatory, zones $+18^{\circ}$, $+20^{\circ}$ (10 charts); Toulouse Observatory, zones $+5^{\circ}$, $+7^{\circ}$, $+9^{\circ}$ (8 charts).
 Astrographic Catalogue: Hyderabad, Nizamiah Observatory, Vol. 6, Dec. -22° .
 A. Kriloff, Russian translation of Isaac Newton's *Principia*, Books I., II., III.
Science Abstracts, Vols. 1 to 9 (presented by Major Levin).

Celestial Photographs.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to the Fellows:—

R.A.S.
 Ref. No.

395. Orion Nebula, 1920 Dec., 30-in. Refl., Exposures 40 min. and 47 min., through colour screens, Reynolds, Helwan.
396. Sunspot Spectrum, 1918 Dec. 25, Spot approaching W. limb, Grating Spectrograph, Evershed.
397. Spectrum of Chromosphere (ultra-violet), 1926 Jan. 14, Sumatra, Stratton and Davidson.
398. Spectrum of Corona (ultra-violet), 1926 Jan. 14, Sumatra, Stratton and Davidson.
399. Jupiter, 1926 Oct. 2, Exposures in ultra-violet and infra-red, Wright, Lick Observatory.
400. Nebecula Minor, 1898 Nov. 10, 24-in. Bruce doublet, Harvard (Arequipa).
401. Orion Nebulosity, 1927 Jan. 6, 3-in. lens, f.l. 21-in., Exp. 5 hours, Ross, Yerkes.

The Effect of Collisions on Monochromatic Radiative Equilibrium.

By E. A. Milne.

1. In two recent papers* I have dwelt on the circumstance that as we proceed outwards from the interior of a star to the highest levels of the stellar atmosphere we find a transition from local thermodynamic equilibrium to monochromatic radiative equilibrium. It is the purpose of this paper to trace this transition in detail as far as regards the transfer of radiation and to establish an equation already used in a previous paper.†

We shall prove that (at least in the case which it is possible to analyse fully) as the density tends to zero, *i.e.* towards the outside of a star, absorption and emission correspond to monochromatic radiative equilibrium: whilst as the density tends to infinity, *i.e.* towards the interior of the star, absorption and emission correspond to local thermodynamic equilibrium. The transition is due to the changing effect of collisions in exciting and de-exciting atoms.

2. It is first necessary to define our terms. The matter in the neighbourhood of a point P will be said to be in *local thermodynamic equilibrium* at temperature T when its emission of radiation is the same as if it were in strict thermodynamic equilibrium at temperature T, *i.e.* is the same as if it were enclosed in a uniform-temperature enclosure at temperature T. Then Kirchhoff's law holds strictly. Let $4\pi\epsilon_\nu$ be the total emission of radiation within the range $(\nu, \nu + d\nu)$ during dt , per unit mass. Let $B_\nu = (2h\nu^3/c^2)(e^{h\nu/kT} - 1)^{-1}$ be Kirchhoff's universal function, the intensity of ν -radiation in a uniform-temperature enclosure at temperature T. Let k_ν be the coefficient of absorption. Then Kirchhoff's law states that

$$\epsilon_\nu = k_\nu B_\nu \quad . \quad . \quad . \quad . \quad (1)$$

Now consider the transmission of radiation through material which is in local thermodynamic equilibrium. Let I_ν be the intensity of ν -radiation in a direction given by the element of length ds . Then by the usual argument the equation of transfer is

$$\frac{dI_\nu}{\rho ds} = -k_\nu I_\nu + \epsilon_\nu \quad . \quad . \quad . \quad . \quad (2)$$

using (1), this becomes

$$\frac{dI_\nu}{\rho ds} = -k_\nu I_\nu + k_\nu B_\nu \quad . \quad . \quad . \quad . \quad (3)$$

3. We proceed now to monochromatic radiative equilibrium. The matter in the neighbourhood of a point P is said to be in *monochromatic radiative equilibrium* in respect of ν -radiation when it

* *M.N.*, **87**, 697, 1927; **88**, 188, 1928.

† *Loc. cit.* (first paper), p. 702, equation (22).

re-emits all the ν -radiation which it absorbs. The total absorption per unit mass is

$$k_{\nu} \int I_{\nu} d\omega,$$

where I_{ν} is the incident intensity of ν -radiation, the integration being taken over the complete solid angle; I_{ν} is of course a function of direction at P. Then the equation of transfer of ν -radiation is found to be

$$\frac{dI_{\nu}}{\rho ds} = -k_{\nu} I_{\nu} + k_{\nu} \int I_{\nu} \frac{d\omega}{4\pi} \quad . \quad . \quad . \quad (4)$$

assuming the absorbed radiation is re-emitted uniformly in direction.

Equations (3) and (4) are of course the classical equations of Schuster and Schwarzschild.

4. We now propose to consider the equation of transfer of radiation when we remove the restriction to either local thermodynamic equilibrium or monochromatic radiative equilibrium. We shall impose only the condition that the state at the point P is a steady state as regards the distribution of the atoms at P amongst their stationary states. We shall show, under such further restrictions as seem to be convenient, that at low densities the equation of transfer tends to the form (4) and that at high densities it tends to the form (3).

We consider the case of an assembly containing atoms of a given species supposed capable of two and only two stationary states. The assembly may contain any number of atoms of other species, also any number of ions and electrons. We refer to the state of lower energy as the normal state (state 1), that of higher energy as the excited state (state 2). Let n_1 be the number of atoms per cm.³ in state 1, n_2 the number in state 2, at the point P. Let $A_{2 \rightarrow 1}$, $B_{1 \rightarrow 2}$, $B_{2 \rightarrow 1}$ be the usual Einstein probability coefficients, defined with reference to the *intensity* of radiation (not the *density*).^{*} If q_1 , q_2 are the statistical weights of the states 1 and 2, we have the Einstein relations

$$\frac{A_{2 \rightarrow 1}}{B_{1 \rightarrow 2}} = \frac{2h\nu^3}{c^2} \frac{q_1}{q_2} \quad . \quad . \quad . \quad (5)$$

$$\frac{B_{2 \rightarrow 1}}{B_{1 \rightarrow 2}} = \frac{q_1}{q_2} \quad . \quad . \quad . \quad (6)$$

It is convenient to abbreviate by writing

$$\sigma = 2h\nu^3/c^2 \quad . \quad . \quad . \quad (7)$$

^{*} This means that, for instance, $B_{1 \rightarrow 2} I_{\nu} dt$ is the probability that an atom will absorb during dt , under the influence of isotropic radiation of intensity I_{ν} . Einstein and most other writers have used a coefficient which we may call $B'_{1 \rightarrow 2}$, such that $B'_{1 \rightarrow 2} \rho_{\nu} dt$ is the probability in question, where ρ_{ν} is the density of ν -radiation. For isotropic radiation, $\rho_{\nu} = 4\pi I_{\nu}/c$, so that $B_{1 \rightarrow 2} = (4\pi/c) B'_{1 \rightarrow 2}$. It appears to the writer that $B_{1 \rightarrow 2}$ is much more convenient to use than $B'_{1 \rightarrow 2}$ when we wish to consider extensions to non-isotropic fields.

Let $\alpha(\nu)$ be the atomic absorption coefficient for an atom in the state 1. Then it is easily shown* that approximately

$$\int \alpha(\nu) d\nu = \frac{B_{1 \rightarrow 2} h\nu}{4\pi} \quad (8)$$

the integral being taken through the narrow range of frequencies within which the atom can absorb. To simplify our work we shall represent this as uniform absorption within an interval $\Delta\nu$, and we shall accordingly write

$$\alpha_\nu \Delta\nu = \frac{B_{1 \rightarrow 2} h\nu}{4\pi} \quad (9)$$

where α_ν is the mean atomic absorption coefficient in $(\nu - \frac{1}{2}\Delta\nu, \nu + \frac{1}{2}\Delta\nu)$.

We next introduce further probability coefficients † $b_{1 \rightarrow 2}$, $a_{1 \rightarrow 2}$ to take account of the effect of collisions in exciting and de-exciting atoms in the states 1 and 2 respectively. Let $b_{1 \rightarrow 2} dt$ be the probability that an atom of the kind in question, in state 1, is excited to state 2 by a collision with any atom, ion, or electron present in the assembly, during dt . Let $a_{2 \rightarrow 1} dt$ be the probability that an atom in state 2 is de-excited to state 1 by a collision, during dt . In the first type of collision, known as "collisions of the first kind" or inelastic collisions, the energy of excitation $h\nu$ is derived from the kinetic energy of the colliding particles. In the second type of collision, known as "collisions of the second kind" or super-elastic collisions, the colliding particles rebound with additional kinetic energy $h\nu$.

The coefficients $A_{2 \rightarrow 1}$, $B_{1 \rightarrow 2}$, $B_{2 \rightarrow 1}$ are atomic constants independent of temperature. The coefficients $a_{2 \rightarrow 1}$, $b_{1 \rightarrow 2}$ are, on the contrary, functions of the temperature. Further, they depend on the total number of particles present, and on their kinds. For a given composition, they will be proportional to the total density. As regards dependence on the temperature, there is strictly speaking no temperature, since we are considering a state of the most general kind, exposed to any kind of external radiation. There will, however, be a certain distribution of velocity amongst the particles present, and this distribution we may assume to be of Maxwellian form. In that case it will mimic the distribution in thermodynamic equilibrium at some temperature T . This T is, as regards the assembly under consideration, merely a parameter characterising the velocity-distribution. It is, however, the "temperature" of the assembly as measured by a thermometer whose bulb was enclosed in a perfectly conducting sheath; for such a thermometer, being screened from radiation by the perfectly

* *M.N.*, **85**, 118, 1924. The same relation has been given by Tolman (*Phys. Rev.*, **23**, 693, 1924) and others.

† R. H. Fowler, *Phil. Mag.*, **47**, 257, 1924. Cf. also Eddington, *Internal Constitution of the Stars*, p. 339 (1926), whose analysis suggested the present treatment. The results are, however, somewhat different. Eddington assumed monochromatic radiative equilibrium in the continuous spectrum adjacent to an absorption line. The present treatment will show that this assumption is not justifiable, as the continuous spectrum must be produced in a region of higher density than the absorption line.

conducting and therefore perfectly reflecting sheath, would simply measure the intensity of the molecular bombardment to which it was subjected.

According to the principle of detailed balancing, if the assembly were in thermodynamic equilibrium we should have super-elastic collisions exactly balancing inelastic collisions. The condition for this is

$$n_1 b_{1 \rightarrow 2} = n_2 a_{2 \rightarrow 1} \quad (10)$$

But in thermodynamic equilibrium at temperature T we have by Boltzmann's relation

$$\frac{n_2}{n_1} = \frac{q_2}{q_1} e^{-h\nu/kT} \quad (11)$$

Hence in thermodynamic equilibrium at temperature T we have

$$\frac{b_{1 \rightarrow 2}}{a_{2 \rightarrow 1}} = \frac{q_2}{q_1} e^{-h\nu/kT} \quad (12)$$

This equation must, however, subsist in virtue of the velocity distribution of the particles present. Hence, although we have proved it in the first instance only for thermodynamic equilibrium, it must also hold whenever the velocity-distribution may be described by a parameter-temperature T . The ratio $\frac{b_{1 \rightarrow 2}}{a_{2 \rightarrow 1}}$ is of course independent of the density.

We accordingly take (12) as holding in general.

As regards the separate dependence of $b_{1 \rightarrow 2}$ and $a_{2 \rightarrow 1}$ on the temperature, it is clear on general grounds that $b_{1 \rightarrow 2}$ will largely depend on the numbers of particles of different kinds endowed with a kinetic energy exceeding $h\nu$. Thus $b_{1 \rightarrow 2}$ will contain a factor $e^{-h\nu/kT}$; its other factors, and also $a_{2 \rightarrow 1}$ will vary much more slowly with the temperature.*

5. Now consider the transfer of radiant energy through the system. Take a thin slab, of area dS and thickness ds . The number of quanta of ν -radiation entering the slab in directions inclined within a solid angle $d\omega$ normal to the slab is

$$\frac{I_\nu \Delta \nu d\omega dS}{h\nu},$$

and the number leaving in the same direction is

$$\frac{(I_\nu + dI_\nu) \Delta \nu d\omega dS}{h\nu}.$$

The excess of the latter number over the former, in the steady state, must be equal to the excess of emissions of ν -quanta from the matter inside the slab in directions included in $d\omega$ over the number of similar absorptions. The number of such emissions is

$$n_2 ds dS (A_{2 \rightarrow 1} + B_{2 \rightarrow 1} I_\nu) d\omega / 4\pi,$$

* Cf. R. H. Fowler, *loc. cit.*

and the number of such absorptions is

$$n_1 ds dS B_{1 \rightarrow 2} I_\nu d\omega / 4\pi.$$

Hence

$$\frac{dI_\nu \Delta\nu}{h\nu ds} = \frac{1}{4\pi} [n_2(A_{2 \rightarrow 1} + B_{2 \rightarrow 1} I_\nu) - n_1 B_{1 \rightarrow 2} I_\nu].$$

Using (5), (6), (7), and (9), this gives

$$\frac{dI_\nu}{\alpha_\nu ds} = -n_1 I_\nu + \frac{q_1}{q_2} n_2 (\sigma + I_\nu) \quad (13)$$

This is the equation of transfer of radiation.

6. By way of parenthesis, we shall now verify that (13) reduces to (3) when the state reduces to strict thermodynamic equilibrium. The two will be identical provided

$$\rho k_\nu = \alpha_\nu \left(n_1 - n_2 \frac{q_1}{q_2} \right) \quad (14)$$

and

$$\rho k_\nu B_\nu = \alpha_\nu \frac{q_1}{q_2} n_2 \sigma \quad (15)$$

Equation (14) provides the relation connecting the mass-absorption coefficient for the whole set of atoms under consideration and the atomic absorption coefficient. Dividing (15) by (14) we get

$$B_\nu = \frac{n_2(q_1/q_2)}{n_1 - n_2(q_1/q_2)} \sigma \quad (16)$$

But in thermodynamic equilibrium (11) holds. Substituting in (16), we get

$$B_\nu = \frac{\sigma}{e^{h\nu/kT} - 1} \quad (17)$$

This is precisely Planck's law, as it should be. Thus (13) includes (3) as a special case. On our assumptions, the mode of transfer of radiation is completely known when n_1 and n_2 are known as functions of position.

7. We now write down the condition that the state is steady. We simply state that the number of transitions $1 \rightarrow 2$ is equal to the number $2 \rightarrow 1$. This gives

$$n_1 \left[B_{1 \rightarrow 2} \int I_\nu \frac{d\omega}{4\pi} + b_{1 \rightarrow 2} \right] = n_2 \left[A_{2 \rightarrow 1} + B_{2 \rightarrow 1} \int I_\nu \frac{d\omega}{4\pi} + a_{2 \rightarrow 1} \right] \quad (18)$$

Write

$$\epsilon = \frac{b_{1 \rightarrow 2}}{B_{1 \rightarrow 2}},$$

and use relations (5), (6), and (12). Then (18) becomes

$$\left(n_1 - n_2 \frac{q_1}{q_2} \right) \int I_\nu \frac{d\omega}{4\pi} - n_2 \frac{q_1}{q_2} \sigma + \epsilon \left[n_1 - n_2 \frac{q_1}{q_2} e^{h\nu/kT} \right] = 0 \quad (19)$$

From what has gone before, the coefficient ϵ is proportional to the total density, and depends on the temperature principally through a factor $e^{-h\nu/kT}$.

8. From (19) we find

$$\frac{n_2(q_1/q_2)}{n_1 - n_2(q_1/q_2)} = \frac{\int I_\nu(d\omega/4\pi) + \epsilon}{\sigma + \epsilon(e^{h\nu/kT} - 1)} \quad (20)$$

Introduce this in (13). It becomes

$$\frac{dI_\nu}{\alpha_\nu ds} = \left(n_1 - n_2 \frac{q_1}{q_2} \right) \left[-I_\nu + \frac{\int I_\nu(d\omega/4\pi) + \epsilon}{1 + (e^{h\nu/kT} - 1)(\epsilon/\sigma)} \right] \quad (21)$$

Introduce the optical thickness τ_ν for ν -radiation, defined by

$$\alpha_\nu \left(n_1 - n_2 \frac{q_1}{q_2} \right) ds = d\tau_\nu = k_\nu \rho ds \quad (22)$$

Further, write

$$\eta = \frac{\epsilon}{B_\nu(\tau)} = \frac{b_{1 \rightarrow 2}}{B_{1 \rightarrow 2}} \frac{e^{h\nu/kT} - 1}{\sigma} \quad (23)$$

Then (21) becomes

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{\int I_\nu(d\omega/4\pi) + \eta B_\nu(\tau)}{1 + \eta} \quad (24)$$

This is the form taken by the equation of transfer when account is taken of the existence of a steady state.

9. We have seen that approximately $\epsilon \propto \rho e^{-h\nu/kT}$. Hence from (23), except for very low temperatures, η is approximately independent of temperature and proportional to the total density. Hence if we apply (24) to a star we see that η is zero at the boundary and increases towards the interior.

As $\eta \rightarrow 0$, (24) tends to the form

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \int I_\nu \frac{d\omega}{4\pi} \quad (25)$$

This is identical with (4), the equation of transfer for monochromatic radiative equilibrium.

As $\eta \rightarrow \infty$, (24) tends to the form

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(\tau) \quad (26)$$

This is identical with (3), the equation of transfer for local thermodynamic equilibrium at temperature T . In (24), we recall, T is simply the parameter-temperature fixing the velocity-distribution.

We infer that as we penetrate a star from the boundary inwards, we begin with monochromatic radiative equilibrium and pass continuously to local thermodynamic equilibrium. The higher the density, that is, the more the atoms are bombarded by other atoms, the more the atoms emit according to Kirchhoff's law.

10. Yet another interpretation can be given to (24). Following

Schuster, let us suppose the material to possess an absorption coefficient k_ν and a scattering coefficient s_ν . Then the equation of transfer is

$$\frac{dI_\nu}{\rho ds} = -(k_\nu + s_\nu)I_\nu + s_\nu \int I_\nu \frac{d\omega}{4\pi} + k_\nu B_\nu \quad (27)$$

This is identical with (24) if we put

$$\eta = k_\nu/s_\nu, \quad d\tau_\nu = (k_\nu + s_\nu)\rho ds \quad (28)$$

Thus our result is equivalent to the statement that near the boundary of the star the dominant feature is scattering without change of frequency, whilst the further we penetrate the star the more we encounter regions where absorbed radiation is re-emitted in frequencies depending on the local temperature, *i.e.* on the local velocity-distribution.

11. It is of interest to attempt a rough numerical evaluation of the quantity η . To see its order of magnitude, let us confine attention to the electronic collisions.

Let N_e be the number of electrons per unit volume, m the mass of the electron. In a Maxwellian distribution at temperature T , the number of electrons moving with velocity between c and $c + dc$ with directions inside $d\omega$ is

$$N_e \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} e^{-\frac{1}{2}mv^2/kT} v^2 dv d\omega.$$

Hence the number with velocities exceeding v_0 colliding with a stationary sphere of radius r in time dt is

$$4\pi N_e \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} \pi r^2 dt \int_{v_0}^{\infty} e^{-\frac{1}{2}mv^2/kT} v^3 dv.$$

Putting $\frac{1}{2}mv_0^2 = h\nu$, this comes to

$$N_e r^2 \left(\frac{8\pi kT}{m} \right)^{\frac{1}{2}} \left(1 + \frac{h\nu}{kT} \right) e^{-h\nu/kT} dt.$$

Let the fraction f of these be effective in exciting atoms from state 1 to state 2. Then we have

$$b_{1 \rightarrow 2} = fr^2 N_e \left(\frac{8\pi kT}{m} \right)^{\frac{1}{2}} \left(1 + \frac{h\nu}{kT} \right) e^{-h\nu/kT}.$$

For $B_{1 \rightarrow 2}$ we may adopt the value *

$$B_{1 \rightarrow 2} = \frac{4\pi}{c} \frac{\pi e^2}{mh\nu},$$

where e is the charge on the electron. This is equivalent to adopting the theoretical value for the mean life of the excited atom.

We have then from (23)

$$\eta = \frac{mc^3}{8\pi^2 r^2 e^2} fr^2 N_e \left(\frac{8\pi kT}{m} \right)^{\frac{1}{2}} \left(1 + \frac{h\nu}{kT} \right) (1 - e^{-h\nu/kT}) \quad (29)$$

We observe that, in accordance with the general line of argument above, η varies only slowly with T and is proportional to the electron density.

From this formula let us find the partial electron pressure $P_e = N_e kT$ when η is unity.

We find

$$(P_e)_{\eta=1} = \frac{8\pi^2\nu^2 e^2}{mc^3} \cdot \frac{1}{f r^2} \left(\frac{mkT}{8\pi} \right)^{\frac{1}{2}} \left(1 + \frac{h\nu}{kT} \right)^{-1} \left(1 - e^{-h\nu/kT} \right)^{-1} \quad (30)$$

The value of f for ionisation collisions has been studied by R. H. Fowler (*loc. cit.*). It appeared that the effective target area was of the order of the cross-section of the atom. Let us assume something of the same kind is true for excitation collisions, and put as regards order of magnitude $f = 1$, $r = 1.5 \times 10^{-8}$ cm. We will further take $T = 6000^\circ$, and take for ν the mean frequency of the H and K lines of Ca^+ . With $e = 4.774 \times 10^{-4}$ E.S.U., $m = 9.00 \times 10^{-28}$ gram, $h = 6.55 \times 10^{-27}$ we find

$$(P_e)_{\eta=1} = 1.46 \times 10^3 \text{ dyne cm.}^{-2} = 1.46 \times 10^{-3} \text{ atmos.}$$

If P_e in the reversing layer of a star is of the order of 10^{-4} atmos., it would follow from this rough calculation that in the reversing layer of a star η is about $\frac{1}{10}$. As we have neglected the effects of collisions other than electronic collisions, η will be rather higher than this. If, however, we adopt a value of the order $\eta = \frac{1}{10}$ for the reversing layer of a star, it would follow that the reversing layer is very close to monochromatic radiative equilibrium. In the photospheric layers we shall have a transition from monochromatic radiative equilibrium to local thermodynamic equilibrium as we go inwards; for $P_e = 10^{-2}$ atmos. we should have $\eta = 10$, which practically corresponds to local thermodynamic equilibrium.

12. Equation (24) has been deduced on special assumptions concerning the atom in question, namely, that it exists in only two stationary states. A difference will be introduced if cyclic processes of the type $1 \rightarrow 3$, $3 \rightarrow 2$, $2 \rightarrow 1$ occur, since this will result in radiation being transformed in frequency. On the other hand, the parallelism of (24) with Schuster's form (27) suggests that (24) has a wider validity than its mode of derivation indicates; it contains in fact no explicit reference to the stationary states of the atom, except in the parameter η .

13. If we assume (24) to be of universal validity for all frequencies we have potentially an equation from which we can deduce the emergent intensity in the whole spectrum, both inside the absorption lines and between them, once we know T and η as functions of the optical depth. Schuster's famous investigation* corresponds in effect to adopting a definite gradient for T and taking $\eta = \text{constant}$. If our arguments are right, this cannot give a good representation of the true state of affairs, since as we have seen η increases inwards from zero to some very large number. Unfortunately it does not seem possible at present to make progress with (24) by using a variable η , say $\eta \propto \rho$. What,

* *Astrophys. Journ.*, 21, 1, 1905.

however, I have attempted to do in two recent papers is to assume $\eta = 0$ on one side of a certain plane, $\eta = \infty$ on the other. This can be at best only a crude representation, but the solution obtained in this way may be expected to exhibit some of the features of the true solution.

According to the above rough numerical estimate, monochromatic radiative equilibrium sets in at $P_e = 10^{-3}$ atmos., as we pass outwards from the photosphere. In the two recent papers mentioned I have assumed demarcation between $\eta = 0$ and $\eta = \infty$ might be taken as dividing the chromosphere and upper reversing layer from the lower reversing layer and photospheric layers. It would now appear that I took this plane too high up, at too low a pressure, and some revision * may be required. Selective radiation-pressure may be appreciable at greater depths in the reversing layer than I estimated.

14. It ought to be mentioned that as we have confined ourselves in this paper to the emission and absorption of radiation by transitions between stationary states, the analysis only refers to the atoms producing the core of the absorption line. It has no bearing on the question of wings and line-widths. We have simply shown that at reversing layer pressures, and lower pressures, collisions do not interfere with monochromatic radiative equilibrium, whilst at photospheric pressures the effect of collisions is to cause a redistribution of intensity in a given frequency so as to make it approximate to that given by Kirchhoff's law. In discussing the formation of a Fraunhofer line, we are bound to take into account both regions. For the photospheric layers themselves will play some part in "reversing" the line. Any calcium, for instance, present in the layers which give an appreciable continuous spectrum must begin to form an absorption line, owing to its absorption coefficient being higher than the general coefficient of absorption. Above the photospheric layers the general coefficient of absorption decreases to zero, and we have layers of gas transparent except in line frequencies; here the lines are deepened.

15. If these suggestions correspond in any way with actuality, they have a general bearing on the theoretical calculation of the profiles of absorption-line energy-curves. Dr. Minnaert, † in his recent important work on the profiles of absorption-curves in the sun's Fraunhofer spectrum, has compared his results with calculations based on the formula

$$I_\nu = \int_0^\infty \epsilon_\nu e^{-k_\nu \tau} k_\nu d\tau.$$

This formula is true generally, if ϵ_ν denotes the coefficient of emission of ν -radiation. But we are only entitled to take $\epsilon_\nu = B_\nu$ (Planck's function) in the region in which local thermodynamic equilibrium holds. Outside these layers, ϵ_ν is no longer given by B_ν . In the centre of the line it is given by

$$\epsilon_\nu = \frac{n_2(q_1/q_2)}{n_1 - n_2(q_1/q_2)} \sigma$$

* Revision of p. 706 of *M.N.*, 87.

† *Zeits. für Phys.*, 45, 618, 1927.

for the case of an atom with only two stationary states; n_2/n_1 may in the outer layer "correspond" to much lower temperatures than those predicted by the theory of radiative equilibrium for the *integrated* radiation, and so ϵ_ν may be very different from $B_\nu(\tau)$. In the wings of the line other considerations such as those brought forward by Stewart* and Unsöld† will be required.

16. The estimate $P_e = 10^{-3}$ atmos. for the layer in which $\eta = 1$ is probably, as already mentioned, an overestimate of pressure. In this connection it may be borne in mind that the collision radius of an atom for excitation by similar atoms may be much larger than its radius as calculated from the kinetic theory of gases. For example, line-broadening for mercury atoms has been found by Orthmann and Pringsheim‡ to correspond to a cross-section some 100 times the gas-kinetic cross-section. On the other hand, line-weakening (de-excitation of excited atoms) seems to correspond to a cross-section only some 10 times the gas-kinetic cross-section. It is possible that the estimate $P_{e(\eta=1)} = 10^{-3}$ atmos. should be reduced to 10^{-4} or 10^{-5} atmos.

17. In conclusion, it may be mentioned that the transition from local thermodynamic equilibrium to monochromatic radiative equilibrium has an important bearing on the change of residual intensity between centre and limb. When strict monochromatic equilibrium holds, the central residual intensity should be cut down by a factor $2/5$ between centre and limb, whereas if local thermodynamic equilibrium held right up to the boundary, as Schwarzschild pointed out, the line at the limb should fade into the continuous background. The fact that lines do not disappear at the limb shows that some measure of monochromatic equilibrium holds for all lines, as predicted by the theory. This is simply the selective scattering inferred by Schwarzschild.

Possibly a better model of a stellar atmosphere could be constructed by fitting on to one another, continuously, three layers, namely, layers for which $\eta = 0$, $\eta = 1$, and $\eta = \infty$, corresponding to increasing depths.

Note on the Theory of the Outer Layers of a Pulsating Star.

By Dr. J. J. M. Reesinck.

(Communicated by the Secretaries.)

1. In *M.N.*, **87**, 414, 1927, the author examined on a very much simplified model the outer layers of a pulsating star. In *M.N.*, **87**, 539, 1927, Professor A. S. Eddington gave a much more exact treatment of the same problem. In a postscript he compares the two investigations, but I think that it may be of some interest to examine

* *Astrophys. Journ.*, **59**, 35, 1924.

† *Zeits. für Phys.*, **44**, 793, 1927; **46**, 765, 1928.

‡ *Ibid.*, **46**, 160, 1927.

somewhat more in detail the points of difference between the two papers.

This will be done in § 2 of the present paper. Eddington's solution for region B will be extended to region C, and in § 3 a more rigorous derivation will be obtained for Eddington's constant of integration A.

Section 4 deals with the determination of the constants p and r in my former paper which, as shown by Eddington in his postscript, was erroneous. The correct solution is given and is shown to be in agreement with Professor Eddington's.

2. Let us, following Eddington, neglect the term $(1 - 4\theta - \theta\sigma)\alpha y/p_0^2$ in his equation (5.9) and put $\lambda = 4$. Introducing the optical depth $x_0 - 1$ by the substitution, $p_0 = g_0(x_0 - 1)/k_0$, we get after some reduction *

$$\frac{d^2}{dx_0^2} \left\{ y(x_0 - 1) \right\} + \frac{dy}{dx_0} - \frac{g_0}{k_0} \frac{i\alpha y}{P} + \frac{2\alpha R_1}{x_0 - 1} = 0 \quad (a)$$

While my equations (13), (14) reduce to

$$\frac{d^2}{dx_0^2} (y x_0) - \frac{g_0}{k} \frac{i\alpha_b y}{P} x_0^{1/4} = 0 \quad (b)$$

Here α_b is the boundary value of Eddington's quantity α .

Concerning the differences between (a) and (b) we remark :

(1) The term dy/dx_0 is missing in (b). The reason is that I have taken $\lambda = 0$ instead of $\lambda = 4$. It is easily obtained in equation (b) by replacing my equation (8) by

$$dx_0 = (1 + 2R_1 - k_1)dx = (1 + 2R_1 - p_1 + 4T_1)dx.$$

(2) As Eddington has taken into account the variability of g_0 , p_1 , and ξ_1 with ξ_0 , while I supposed these quantities to be constant (except, of course, where the variability of ξ_1 has a direct influence on the variation of the density), he obtains the term $2\alpha R_1/(x_0 - 1)$ in (a).

(3) Instead of $\alpha_b x_0^{1/4}$ Eddington takes a mean value α for region B. This is equivalent to taking k_0 directly proportional to the temperature.

(4) The first term in (a) and (b) is different in consequence of Eddington's taking T_0^4 proportional to p_0 . Taking in the derivation of his differential equation T_0^4 proportional to $p_0 x_0/(1 + x_0)$, according to Schwarzschild's theory, the first term of (a) becomes identical with that of (b). The last term in (a), however, remains $2\alpha R_1/(x_0 - 1)$, but except in the outermost layers of region C we may neglect the 1 in the denominator. So we obtain the equation

$$\frac{d^2}{dx_0^2} (y x_0) + \frac{dy}{dx_0} - \frac{g_0}{k_0} \frac{i\alpha y}{P} + \frac{2\alpha R_1}{x_0} = 0 \quad (a')$$

By the substitution $\dagger x_0 = k_0 p_0/g_0$, followed by the substitution (6.1), it transforms of course into (6.3); so the solution of (a') is obtained from (6.5) by the inverse transformations.

* Here g_0 is supposed to be a constant.

† So here the quantity p_0 no longer means the pressure. It is not zero at the boundary.

The solution of (b) is given in my equations (24), (25).

The particular integral $2iR_1(1 + 4/v^2)$ in (6, 5) which is missing in my solution is caused by the term mentioned under (2).^{*} Though this term may be expected to be rather unimportant, it appears that the particular integral grows large for small values of v . This point will be discussed in § 3.

The difference in the order of the Bessel functions is easily explained by the differences mentioned under (1) and (3).

3. To determine the constant A in equation (6.5), Professor Eddington uses for region C the equation :

$$2T_1 + p_0 \frac{dT_1}{dp_0} = \text{const.}$$

After differentiation and transformation to the variables u and v it becomes :

$$\frac{d^2u}{dv^2} + \frac{1}{v} \frac{du}{dv} - \frac{4u}{v^2} = 0,$$

an equation differing from (6.5) only by the want of the terms u and $2iR_1$. The term u contains the storage of heat, and this term is insignificant for small values of v , as is also evident from the equation (6.3) itself. But the term $2iR_1$ is one of the improvements of Eddington's investigation; it is the term that gives rise to the particular integral in (6.5), and as the constant of integration A is chosen so as to neutralise almost completely this particular integral in region C, it might be dangerous to neglect this term when applying the boundary condition.

Moreover, as equation (a') and its solution remain valid up to the boundary, it is not necessary to use a simplified form of it for region C. Therefore we shall apply to this equation the exact boundary condition that at the surface there is no inward stream of energy. In my paper this condition is formulated in the equations (22), (23). Taking into account the variability of k it becomes : †

$$\frac{dy}{dx_0} + y = -\frac{1}{2}R_1.$$

The condition that in Eddington's equation (6.6) the constant b is neither large nor complex ‡ must follow from this equation.

As near the boundary v is very small, it is sufficient to retain in the development of (6.5) only the lowest powers of v . So we get

$$u = -2iR_1 \left(1 + \frac{4}{v^2} \right) \left(1 - \frac{A}{\pi} \right) + \frac{A}{\pi} \cdot \frac{1}{4} v^2 \{ l(\frac{1}{2}v) - \frac{3}{4} + \gamma \},$$

* The term $2i\xi_1$ in (6.3) and (6.5) should have the negative sign.

† If we take into account the variability of p_1 an unimportant term is added.

‡ Though it is not very probable that b is large, it might be the case in a mathematical solution in which the second order terms are neglected. b is irrelevant to the flow of heat but not to the temperature, which itself is one of the observational data.

or neglecting some terms which near the boundary are very small in comparison with the remaining ones : *

$$y = + 2R_1 \frac{k_0^2 P^2}{g_0^2 x_0^2 \alpha} \left(1 - \frac{A}{\pi} \right).$$

So the boundary condition gives :

$$1 - \frac{A}{\pi} = \frac{g_0^2 \alpha}{4k_0^2 P^2}.$$

The right-hand member is of the order of 10^{-6} . So A is only very little different from π ; y is very small everywhere in region C except near the boundary where it takes the form (cf. the end of § 8 of my previous paper) :

$$y = R_1 / 2x_0^2.$$

As might be expected *a priori* the term $2\alpha R_1/x_0$ has no influence on the result. Here we have one of the many cases in which an insignificant change of a differential equation seems to give an important modification of the solution, but actually after applying the boundary condition nearly the same solution is obtained, the constants of integration being determined so as to neutralise almost completely the influence of the change.

To the foregoing developments the objection may be made that the last term of (a) is not $2\alpha R_1/x_0$ but $2\alpha R_1/(x_0 - 1)$ which tends to infinity for $x_0 = 1$. The logarithmic term in (c) is therefore not negligible at the boundary but it becomes infinite ; it is the same as the logarithmic singularity mentioned by Professor Eddington.

The answer to this objection is that there are no terms actually that tend to infinity. The terms taken together by Eddington in $2\alpha R_1/(x_0 - 1)$ take their origin in the first term of (5.4). In (5.6) this equation is differentiated with respect to ξ_0 , and after some reduction a constant mean value is introduced for the terms in ξ_1 . As the equation is divided by $g_0 \rho_0$ a term is obtained which becomes infinite for $x_0 = 1$. From equation (5.4), however, it is evident that the first and second terms of (5.6) must vanish for large values of ξ_0 . So in the exact equations the term in ξ_1 remains small up to the boundary, and equation (a') gives a sufficient approximation.†

4. In § 9 of my previous paper I started from the wrong supposition that my solution becomes valid in region B if the constants A and B are different from zero. Here I overlooked the fact that $Y_n(z) + iJ_n(z)$ tends exponentially to zero for large values of z if the imaginary part of z is negative,‡ and that therefore my conditions (20), (21) are satisfied if the ratio of the constants A and B in (24) is equal to $e^{4\pi i/5}$.§

As my former paper referred to a mathematically possible model

* As v is not zero at the boundary the logarithm and its differential quotient are finite.

† A similar reasoning shows that the neglect of the term $(1 - 4\theta - 0\sigma)\alpha y/p_0^2$ in (5.9) was legitimate.

‡ Cf. Eddington, p. 546 at the top.

§ Cf. Eddington, p. 547.

in which the only serious neglect was that k was taken as a constant, while, on the other hand, the variability of a , neglected by Eddington, is taken into account, it may be of interest to see in what way my equations and conclusions should be altered.

After application of the right condition for the interior my equation (24) becomes :

$$T_1 - T_1(ad) = \text{constant} \times x_0^{-1/2} \{ \exp(2\pi i/5) J_{+4/5}(\frac{8}{5}\sqrt{-aix_0^{5/8}}) - \exp(-2\pi i/5) J_{-4/5}(\frac{8}{5}\sqrt{-aix_0^{5/8}}) \}.$$

Retaining only the first term in the development of the Bessel functions, we obtain for region C :

$$T_1 - T_1(ad) = \text{constant} \times \left\{ \frac{(4/5)^{4/5} a^{2/5} \exp(\pi i/5)}{\Gamma(9/5)} - \frac{1}{x_0} \frac{(4/5)^{-4/5} a^{-2/5} \exp(-\pi i/5)}{\Gamma(1/5)} \right\}$$

Computing the arbitrary constant by (22), (23) we get :

$$T_1 - T_1(ad) = -1, 72a^{4/5} R_1 \exp(2\pi i/5) + \frac{1}{2} \frac{R_1}{x_0}.$$

As a is small, the first term may be neglected, and the solution is almost identical with that of my former paper. The result is in agreement with Professor Eddington's.

On the Construction of Tables by Interpolation.

By L. J. Comrie, M.A., Ph.D. (Plate 6.)

Summary.—The Lagrangian method of interpolating to fixed subdivisions of an interval is briefly described and criticised as being too laborious, and capable of concealing errors in the original data. In the method proposed the end figures of the interpolates are derived by simple 2-figure additions. From these the end figures of the second or third differences are found, and the whole value of this difference inferred. A summation process then yields the complete values of the interpolates. The application of various commercial calculating machines to the different stages of the work is developed.

Introduction.—It is well known that in the construction of large tables the labour of computing each value individually from the fundamental formula is usually reduced considerably by calculating only a limited number of values to act as a framework, and then finding the intervening ones by some process of interpolation. A survey of the literature on the subject indicates that there are two principal methods in vogue for performing this type of interpolation—the Lagrangian and the method of breaking down one of the difference columns of the original or pivotal values to give the corresponding column of the interpolates. The former has been described by Professor Karl Pearson *

* Karl Pearson, *Tracts for Computers*, No. II. "On the Construction of Tables and on Interpolation. Part I. Uni-variate Tables." London, Cambridge University Press, 1920.

and is used in certain statistical circles, while the latter has been used in astronomical circles, particularly by Godward, Newcomb, Bauschinger and Peters, and Brown. It may be remarked, however, that most of the existing tables containing more than about 12 decimals appear to have been computed either value by value or else interpolated by some inherent property of the function itself, *e.g.* $\log xy = \log x + \log y$; $\sin(x+y) = \sin x \cos y + \cos x \sin y$; $e^{x+y} = e^x \times e^y$.

The Lagrangian Method.—This consists of finding and employing coefficients by which successive values . . . $f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3, \dots$ of a function are multiplied to give interpolates lying between f_0 and f_1 . A formula involving n values of the function is called an n -point formula, and is equivalent to the use of a central difference interpolation formula involving differences of order $n - 1$. The most convenient formulæ are those for interpolation to fifths and to halves, so that in the common case of desiring to interpolate to tenths it will probably be found best in the Lagrangian process to make a double application, first breaking down the function to halves, and then the resulting values to fifths. For interpolation to halves we have, for instance, the following 6-, 7- and 8-point formulæ:—

$$\begin{aligned} f_{\cdot 5} &= \frac{1}{256}(3f_{-2} - 25f_{-1} + 150f_0 + 150f_1 - 25f_2 + 3f_3) \\ f_{\cdot 5} &= \frac{1}{1624}(-5f_{-3} + 42f_{-2} - 175f_{-1} + 700f_0 + 525f_1 - 70f_2 + 7f_3) \\ &= \frac{1}{1024}(7f_{-2} - 70f_{-1} + 525f_0 + 700f_1 - 175f_2 + 42f_3 - 5f_4) \\ f_{\cdot 5} &= \frac{1}{2048}(-5f_{-3} + 49f_{-2} - 245f_{-1} + 1225f_0 + 1225f_1 - 245f_2 + 49f_3 - 5f_4) \end{aligned}$$

while the 6-point formula for interpolation to fifths is as follows:—

Coeff. of	$f_{\cdot 2}$	$f_{\cdot 4}$	$f_{\cdot 6}$	$f_{\cdot 8}$
f_{-2}	+·008064	+·011648	+·010752	+·006336
f_{-1}	-·073920	-·099840	-·087360	-·049280
f_0	+·887040	+·698880	+·465920	+·221760
f_1	+·221760	+·465920	+·698880	+·887040
f_2	-·049280	-·087360	-·099840	-·073920
f_3	+·006336	+·010752	+·011648	+·008064

Pearson claims (p. 22) that “the Lagrange formula . . . has marked superiority over difference formulæ when we come to computing tables *de novo*.” With this we do not agree. In comparing the merits of the Lagrangian and central difference formulæ it should be emphasised that no question of relative accuracy enters, for both sets of formulæ are equivalent, and each may be converted algebraically into the other. Pearson’s claim states that (p. 6) “As a rule with modern computing machines it is more rapid and efficient to interpolate between the (original) entries . . . by Lagrangian formulæ using the 4, 6, 8 or 10 adjacent entries as the case may require, rather than by constructing an elaborate system of differences for the (original entries), . . . which at any rate will have to be discarded as of no value for the completed table, and will very probably have to be discarded still earlier, if it turns out that the (original entries) have been selected at too great intervals for effective interpolation.” Further (pp. 6-7), “Eight- and ten-entry

interpolation formulæ . . . represent one continuous process on the machine and have not been found too lengthy for practical use." One further quotation (p. 7) may perhaps be permitted: "Throughout the process (i) interpolated values must be checked by occasional directly calculated values, (ii) the system of differences should be taken, the new differences obtained by each interpolated value as it is entered being a very desirable if not wholly adequate test of the accuracy of the machinings. . . ."

To neglect the differencing of the original entries opens the door at once to the risk of very serious error, for if one of these entries is in error by a small amount, quite a number of the interpolates will be affected, and, provided the error is not too great, the new system of differences will utterly fail to reveal it. To illustrate more forcibly this very real danger, 7-figure logarithms of the numbers 0.6, 0.7 . . . 1.8 were taken, but with log 1.2 or 0.0791812 deliberately altered by 10 units in the last decimal to 0.0791802. The portion from 1.0 to 1.4 was interpolated to tenths, by using first 8-point interpolation to halves, followed by 6-point interpolation to fifths, the formulæ for which have already been quoted. The results are exhibited in Table A, together with their fourth differences, which are seen to be exceptionally smooth,

TABLE A.—*Lagrangian Interpolation with Error at $x=1.20$.*

x	$\log x$	$\Delta^4 v$	True — Computed	x	$\log x$	$\Delta^4 v$	True — Computed
1.00	0.0000000	-1	0	1.20	0.0791802	-1	+10
0.01	0.0043214	-2	0	0.21	0.0827844	+1	+10
0.02	0.0086003	+1	-1	0.22	0.0863589	+1	+9
0.03	0.0128374	-1	-2	0.23	0.0899042	-2	+9
0.04	0.0170335	-2	-2	0.24	0.0934209	0	+8
0.05	0.0211895	0	-2	0.25	0.0969094	0	+6
0.06	0.0253061	0	-2	0.26	0.1003701	+1	+6
0.07	0.0293840	+1	-2	0.27	0.1038034	-1	+3
0.08	0.0334239	-2	-1	0.28	0.1072098	-1	+2
0.09	0.0374266	+1	-1	0.29	0.1105897	+2	0
0.10	0.0413927	-1	0	0.30	0.1139434	-1	0
0.11	0.0453229	+1	+1	0.31	0.1172714	-1	-1
0.12	0.0492178	-2	+2	0.32	0.1205741	+2	-2
0.13	0.0530781	+3	+3	0.33	0.1238518	-3	-2
0.14	0.0569043	-4	+6	0.34	0.1271050	+3	-2
0.15	0.0606972	+3	+6	0.35	0.1303339	-2	-1
0.16	0.0644572	-3	+8	0.36	0.1335390	+1	-1
0.17	0.0681850	+3	+9	0.37	0.1367206	-1	0
0.18	0.0718810	-2	+10	0.38	0.1398791	0	0
0.19	0.0755459	0	+11	0.39	0.1430148	+1	0
1.20	0.0791802	-1	+10	1.40	0.1461280	-1	0

especially when it is recalled that fourth differences which average zero frequently attain ± 6 , and may even legitimately attain ± 7 . No suspicion of the errors in $\log x$, as shown in the last column, would be entertained by examining the differences. What provision can be made against their intrusion? Most emphatically the differencing of the original values, which is a very powerful method of detecting accidental errors. Repetition of the original calculations is the weakest possible check, as errors are liable to be repeated, especially if they are due to errors or misprints in a table that has been employed. The direct calculation of occasional values among the interpolates, insisted on by Pearson, is not always feasible, especially when the original values are obtained by quadrature, and in any case is too elusive, as the value chosen for checking might easily be one where the effect of the error did not exceed ± 1 . It may perhaps be urged that the use of an odd-point formula in both its forms (see the 7-point formula for $f_{.5}$ above) would have yielded slightly discordant results at 1.15 and 1.25. This is true, but the computer would probably attribute the discordance to the use of too few points in his formula, and would resort to the next higher formula. Moreover it seems logical to use a symmetrical or even-point formula for interpolation to halves, although, curiously enough, Pearson uses only the odd-point formulæ for this purpose. In the case of interpolation to fifths, however, the symmetrical formulæ are always used, and it seems that, in these circumstances, the error would remain undetected unless it happened to be revealed by one of the occasional check values.

The objection that the differences may have to be discarded if the original intervals are too great is not really serious. The computer should aim, when calculating his pivotal values, at making differences of a certain order, say the sixth or seventh, negligible. If perchance it should happen in a small portion of his table that one or two more orders of differences are sensible, his process of interpolation should be sufficiently elastic to be able to include these extra differences. Trouble only arises if one "sails too close to the wind."

But condemnation of the Lagrangian process merely on the grounds that the omission of the differencing of the original pivotal values has been suggested would be unsound if it could be shown that the process itself possessed marked advantages over alternative schemes, and the question of whether eight- and ten-entry formulæ are too lengthy for practical use is one on which opinions different from those expressed by Pearson may be held. In this connection it may be noted that the 6-entry formulæ, when used with 7-figure functions, permit about 30 interpolates to be derived per hour (exclusive of their subsequent differencing). A 10-entry formula, which is necessary if eighth differences are sensible, applied to a 10-figure function, would not enable more than 10 or 12 interpolates to be derived per hour. Further, many machines have a capacity of only $8 \times 8 \times 16$ or $9 \times 9 \times 18$, and even if a machine like the Monroe is available with capacity $10 \times 10 \times 20$ its results are rendered doubtful by the fact that the tens transmission mechanism will not operate through more than 13 figures at a time.

The only machines that carry throughout a carriage of 20 figures are the Brunsviga Triplex and MD. models, and the Millionaire.

It is submitted, therefore, that the Lagrangian method is too laborious, that it does not lend itself readily to ordinary machines in the case of a many-figure function, and that, when conducted in the manner described by Professor Pearson, errors are liable to pass undetected. It involves the use of large quantities and large coefficients, whereas ordinary interpolation involves progressively diminishing quantities and progressively diminishing coefficients.

Interpolation by Preparation of One of the Difference Columns of the Interpolates.—This is the method that has been in use as an alternative to the Lagrangian process, but, as ordinarily employed, it lacks continuity, in that adjustments must be made after the interpolation of each interval. The frequency with which these adjustments have to be made is sometimes reduced by using extra fictitious decimals, a procedure which, whilst throwing more work on the computer, and taxing the capacity of the machine, still fails to make the process strictly continuous.

Method Proposed.—This is a development of the method just mentioned, designed to avoid the disadvantages referred to, to give almost if not quite complete security against accidental error, and at the same time to make the labour as light as possible, and especially to seize the advantage accruing when the final stage virtually begins with the smallest quantity involved, *i.e.* one of the higher order differences of the interpolates, and proceeds, with mechanical aids, to the heaviest quantity involved.

The fundamental idea is that the end figure only of each interpolate should be found first. By differencing these figures the end figures of successive differences are formed, and a stage is soon reached, which is arranged to be not later than the third difference, at which it is possible to infer the whole of the difference when its end figure is known. The lower order differences, and the interpolates of the function itself, are then produced by continuous summation.

The method, as developed, applies particularly to interpolation to tenths, but is instantly converted into a method for fifths by omitting the figures for $n = \cdot 1, \cdot 3, \cdot 5, \cdot 7$ and $\cdot 9$. By repeating the process the original intervals may be 25, 50 or 100 times the final interval, although it has been found in practice that an original interval greater than 10 times the final interval is seldom practicable. As a general rule the original seventh differences should not exceed 1000, otherwise it becomes difficult to detect errors by differencing.

The notation employed, and the scheme of differences involved, are as follows :—

Function	Differences.							
	First	Second	Third	Fourth	Fifth	Sixth	Seventh	Eighth
f_0		Δ_0''		Δ_0^{iv}		Δ_0^{vi}		Δ_0^{viii}
	$\Delta_{\frac{1}{2}}'$		$\Delta_{\frac{1}{2}}'''$		$\Delta_{\frac{1}{2}}^v$		$\Delta_{\frac{1}{2}}^{vii}$	
f_1		Δ_1''		Δ_1^{iv}		Δ_1^{vi}		Δ_1^{viii}

For convenience the subscript $\frac{1}{2}$ will be omitted, as this cannot cause any confusion. The interpolation formula known as Bessel's, but really discovered by Newton, is

$$f_n = f_0 + n\Delta' + \frac{n(n-1)}{2.2!}(\Delta_0'' + \Delta_1'') + \frac{n(n-1)(n-\frac{1}{2})}{3!}\Delta''' \\ + \frac{(n+1)(n)(n-1)(n-2)}{2.4!}(\Delta_0^{iv} + \Delta_1^{iv}) \\ + \frac{(n+1)(n)(n-1)(n-2)(n-\frac{1}{2})}{5!}\Delta^v \\ + \frac{(n+2)(n+1)(n)(n-1)(n-2)(n-3)}{2.6!}(\Delta_0^{vi} + \Delta_1^{vi}) + \text{etc.}$$

The reasons for the choice of this formula will be given later. The values of these coefficients for $n = .1, .2, . . .$ are shown in Table B. Those up to and including Δ^v are exact, while the remainder are rounded, except those for fifths in $\Delta_0^{vi} + \Delta_1^{vi}$.

TABLE B.—*Coefficients of Bessel's Formula.*

n	$\Delta_0'' + \Delta_1''$	Δ'''	$\Delta_0^{iv} + \Delta_1^{iv}$	Δ^v	$\Delta_0^{vi} + \Delta_1^{vi}$
0.1	-.0225	+0.006	+0.00391875	-0.0006270	-0.0007955
.2	-.0400	+ .008	+ .00720000	- .0008640	- .0014784
.3	-.0525	+ .007	+ .00966875	- .0007735	- .0020014
.4	-.0600	+ .004	+ .01120000	- .0004480	- .0023296
.5	-.0625	.000	+ .01171875	.0000000	- .0024414
.6	-.0600	-.004	+ .01120000	+ .0004480	- .0023296
.7	-.0525	-.007	+ .00966875	+ .0007735	- .0020014
.8	-.0400	-.008	+ .00720000	+ .0008640	- .0014784
.9	-.0225	-.006	+ .00391875	+ .0006270	- .0007955

n	Δ^{vii}	$\Delta_0^{viii} + \Delta_1^{viii}$	Δ^{ix}
0.1	+0.000091	+0.000172	-0.000015
.2	+ .000127	+ .000321	- .000021
.3	+ .000114	+ .000436	- .000019
.4	+ .000067	+ .000509	- .000011
.5	.000000	+ .000534	.000000
.6	- .000067	+ .000509	+ .000011
.7	- .000114	+ .000436	+ .000019
.8	- .000127	+ .000321	+ .000021
.9	- .000091	+ .000172	+ .000015

The material chosen for explanation and illustration of the method is given in Table C, the italic figures being the double sums of even differences, usually entered in red.

TABLE C.—*Original Values and Differences.*

x	$\log x=f$	Δ'	Δ''	Δ'''	Δ^{IV}	Δ^V	Δ^{VI}	Δ^{VII}	Δ^{VIII}
1.0	0.0000000		-43648		-2696		-578		-295
		+413927	79690	+7606	4523	+869	912	+244	372
1.1	.0413927		36042		1827		334		- 77
		377885	66305	5779	3119	535	501	167	217
1.2	.0791812		30263		1292		167		-140
		347622	56039	4487	2216	368	307	27	
1.3	.1139434		25776		924		140		+ 62
		321846	47989	3563	1620	228	191	89	
1.4	.1461280		22213		696		51		- 99

If the calculation for $x = 1.23$ were done in full it would be as shown in the accompanying table. This involves multiplication of each difference in full by its coefficient in full, writing the complete value of the product up to the $p + 1$ th figure if the original function contains p figures, and finally mixed addition and subtraction. It is clear that this labour will be greatly reduced by dealing only with the figures shown in bold type, replacing the multiplications by direct entries from simple tables, and replacing all subtractions by additions.

The work for each division of an interval is arranged in ten vertical columns, the first containing (if it is desired to record them) the various arguments, which depend on the differences, and the other nine the p th and $p + 1$ th figures of the nine interpolates. The contributions of f_0 and Δ' can be combined with the aid of a machine * by putting ten times the last two figures of f_0 in the product register, and the last two figures of Δ' on the setting levers or keyboard. As each turn of the handle is made (backward turns if Δ' has the opposite sign to f_0) the last two figures in the product register are recorded at the head of the nine columns. It is easy to make two turns, memorise four figures, turn again without looking at the machine, write down the four figures memorised, and then glance at the machine to pick up the next two figures. On reaching f_1 the setting of the keyboard is changed to the last two figures of the new Δ' , and the work proceeds. The multiplier register (which should have tens transmission) can be made to show the argument at each stage. This leads to the first line of the following table, the argument column being devoted to the value of the argument x for f_0 .

* It is fair to assume possession of a calculating machine. When a good machine can be bought for £50 or £60 the constructor of tables on a large scale cannot afford to be without one.

Arg.	Contribution to								
	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.2	42	64	86	08	30	52	74	96	18
-39	09	16	20	23	24	23	20	16	09
+4450	67	56	12	78	00	22	88	44	33
-2150	16	45	92	59	48	59	92	45	16
Sums *	34	81	10	68	02	56	74	01	76

The value 86 for $n = .3$ is seen to correspond with the sum of the items 20 + 66 in the example for $x = 1.23$.

Turning now to the effect of $\Delta_0'' + \Delta_1''$ it will be noticed, first, that the coefficients are symmetrical about $n = .5$, and secondly, that 4000 times any of these coefficients is always a multiple of 10. Hence in making a table for the contribution of the second difference it is not necessary to tabulate more than 5 values for each argument, or to continue the table beyond 4000. The tables employed, which bear the same numbers as the differences they cater for, are not given in full, but are illustrated in Table D by specimens. The unit employed is the $p + 1$ th figure. Entries which would normally be negative are replaced by their complements to 100; this avoids all subtraction. Thus if $\Delta_0'' + \Delta_1'' = 12$ its contribution to .1 or .9 would be $12 \times -.0225$ or $-.27$. In the units employed this is -3 , and Table II shows 97. Since the "period" of the table is 4000, positive arguments are arranged on the left, and complements to 4000 as negative arguments on the right. A further shortening of the table is made possible by tabulating for every third value only of the argument. The maximum difference between two consecutive tabular values is 2, in which case the intermediate value is taken for an intermediate argument, otherwise the value for the nearest tabular argument is taken. The whole table occupies 14 foolscap sheets. The second figure repeats after 400, 50, 400, 50 and 16 entries respectively, which implies, of course, that it increases by 5 after 200, 25, 200, 25 and 8 entries respectively. The value -56039 of $\Delta_0'' + \Delta_1''$ occurring in the illustration becomes -39 for the purpose of entering Table II, and it will be seen that 20 is reproduced for $n = .3$.

Examining the coefficients of Δ''' , the period is seen to be 10000, but the table is not carried beyond 5000, as the entry for $5000 + a$ differs from the entry for a only by the addition of ± 50 for $n = .3$ and $n = .7$. As the coefficients for $1 - n$ are those of n reversed in sign, the entries for $1 - n$ will be the complements of those for n . Thus the table is easy to make, and only one argument is needed, for the entries are read in the reverse order for negative arguments. It is sufficient to tabulate for every 10th value of Δ''' . No attempt should be made to interpolate the table. The second figure repeats after 50, 50, 100 and 50 entries respectively; in preparing the table the second figure is first entered, and then the first figure can be written at sight.

* Only the tens and units are written down.

TABLE D.—Specimens of Interpolation Tables.

Table II.—Argument, Remainder after dividing $\Delta_0'' + \Delta_1''$ by 4000.

Positive arguments on left.							Negative arguments on right.						
0	00	00	00	00	00	4000	1692	19	23	12	85	42	2308
3	99	99	98	98	98	3997	1695	19	22	10	83	41	2305
6	99	98	97	96	96	3994	1698	18	21	09	81	39	2302
9	98	96	95	95	94	3991							
12	97	95	94	93	92	3988	3960	09	16	21	24	25	40
							3963	08	15	19	22	23	37
309	30	76	38	15	07	3691							
312	30	75	36	13	05	3688	3999	00	00	01	01	01	1

Table III.—Argument, Last Four Figures of Δ'''

If the argument is negative, read backwards.

0	00	00	00	00	00	00	00	00	00	00
10	01	01	01	00	00	00	99	99	99	99
20	01	02	01	01	00	99	99	98	99	99
30	02	02	02	01	00	99	98	98	98	98
720	43	58	50	29	00	71	50	42	57	57
2510	51	01	76	00	00	00	24	99	49	49
3540	12	83	48	42	00	58	52	17	88	88
4450	67	56	12	78	00	22	88	44	33	33
4990	99	99	49	00	00	00	51	01	01	01

Table IV.—Argument, $\Delta_0^{1v} + \Delta_1^{1v}$

	Argument positive						Argument negative					
0	00	00	00	00	00	00	00	00	00	00	00	00
10	00	01	01	01	01	00	99	99	99	99	99	99
20	01	01	02	02	02	99	99	98	98	98	98	98
30	01	02	03	03	04	99	98	97	97	97	96	96
1580	62	14	53	77	85	38	86	47	23	15	15	15
2150	84	55	08	41	52	16	45	92	59	48	48	48
3020	18	17	92	38	54	82	83	08	62	46	46	46
4350	70	13	21	87	10	30	87	79	13	90	90	90

Table IV cannot be regarded as periodic, and must therefore be made as far as required for both positive and negative arguments unless the computer is content with one table and can take complements to 100 at sight. It is sufficient to tabulate for every 10th value of the argument up to, say, 5000, and then to tabulate the values 10000, 15000, etc., which can be used in combination with values less than 5000 in the case of unusually large arguments.

Table V, if required, would be similar to Table III, but with interval 100, while Table VI would resemble Table IV, but with interval 50. These tables, however, are rendered unnecessary by the fact that, for values of Δ^v or $\Delta_0^{v1} + \Delta_1^{v1}$ less than 10000 it is possible to allow for their effect by a slight variation of the arguments Δ''' and $\Delta_0^{iv} + \Delta_1^{iv}$. Similarly if $\Delta_0^{iv} + \Delta_1^{iv}$ is less than 1000 its use is replaced by a variation of $\Delta_0'' + \Delta_1''$.

Bessel's formula may be written :—

$$f_n = f_0 + n\Delta' + \frac{n(n-1)}{4} \left((\Delta_0'' + \Delta_1'') + \frac{(n+1)(n-2)}{12} (\Delta_0^{iv} + \Delta_1^{iv}) \right) + \frac{n(n-1)(n-\frac{1}{2})}{6} \left(\Delta''' + \frac{(n+1)(n-2)}{20} \Delta^v \right) + \text{etc.}$$

The product $(n+1)(n-2)$ varies from -2.09 to -2.25 so that a constant value may be assumed with a resulting error that will be insensible if $\Delta_0^{iv} + \Delta_1^{iv}$ or Δ^v is not too great. In general, any difference of higher order than the third, if not above a certain limit, may be completely allowed for by multiplying it by a constant factor (always negative) and applying the product to the difference two orders lower. Table E shows these factors, the limiting value of the difference to which they can be applied if an error of about .2 units of the last figure is considered negligible, as well as the coefficients by which the differences would have to be multiplied to give the further correction required to represent Bessel's formula perfectly. A critical table giving multiples of .184 is easily made, whereas for Δ^v and higher order differences the approximate factors $-\frac{1}{9}$, $-\frac{1}{5}$ and $-\frac{1}{7}$ should meet all cases arising in practice.

TABLE E.—*Factors for Reduction of Higher Order Differences.*

Difference	$\Delta_0^{iv} + \Delta_1^{iv}$		Δ^v		$\Delta_0^{v1} + \Delta_1^{v1}$	
Factor	$-\frac{1}{8}$	$-.184$	$-\frac{1}{9}$	$-.108$	$-\frac{1}{5}$	$-.207$
Limit	100	1000	5000	10000	2000	10000

<i>n</i>	<i>Residual Coefficients *</i>					
.1	+·00017	-.00022	+·000040	+·000019	-.000012	+·000016
.2	+·00053	-.00016	+·000025	-.000002	-.000038	+·000012
.3	+·00092	+·00001	+·000004	-.000019	-.000068	·000000
.4	+·00120	+·00016	-.000004	-.000017	-.000090	-.000011
.5	+·00130	+·00022	·000000	·000000	-.000098	-.000016

* These coefficients are, of course, not to be applied. They have been tabulated here merely to show that the use of the given factors for differences less than the limits given is legitimate. Thus if $\Delta_0^{iv} + \Delta_1^{iv} = +1000$, the error introduced by applying $-.184$ to $\Delta_0'' + \Delta_1''$ is $-.22$ units of the last figure at $n = .1$ and $n = .9$, $-.16$ units at $n = .2$ and $n = .8$, etc.

TABLE E.—Factors for Reduction of Higher Order Differences—continued.

Difference	Δ^{vII}		$\Delta_0^{vIII} + \Delta_1^{vIII}$	
Factor	$-\frac{1}{7}$	$-.147$	$-\frac{1}{5}$	$-.218$
Limit	50,000	200,000	4000	100,000

n	Residual Coefficients *			
·1	+·0000013	-·0000010	+·000013	-·0000017
·2	+·0000033	·0000000	+·000025	-·0000013
·3	+·0000039	+·0000010	+·000036	·0000000
·4	+·0000026	+·0000009	+·000043	+·0000013
·5	·0000000	·0000000	+·000046	+·0000018

The factor for Δ^v is more accurately $-.1077$ and that for $\Delta^{vII} = .1466$. The factors and limits for interpolating to fifths are the same as in the table above, with the following exceptions:—

Difference	Δ^v		$\Delta_0^{vI} + \Delta_1^{vI}$	Δ^{vII}
Factor	$-\frac{1}{5}$	$-.109$	$-.207$	$-.147$
Limit	8000	20,000	20,000	300,000

n	Residual Coefficients *			
·2	+·000025	+·000010	+·000012	-·0000006
·4	+·000004	-·000011	-·000011	+·0000006

The factor for Δ^v is more accurately $-.1093$ and that for $\Delta^{vII} = .1473$.

In the illustration the argument Δ''' is to be taken as $+4487 - \frac{1}{5} \cdot 368 = +4446$ or $+4450$ to the nearest tabular argument. The value 12 given by the table for $n = .3$ differs by a turn of the figure only from the value $14 - 3$ or 11 given by Δ''' and Δ^v in the full working. The argument for $\Delta_0^{vI} + \Delta_1^{vI}$ is $-2216 + \frac{1}{5} \cdot 307 = -2150$ to the nearest tabular argument. The value 92 for $n = .3$ agrees exactly with the sum of the values -14 and $+6$ when 100 is added to avoid negative terms.

The final stage of the work is shown in Table F. The figures now known are these shown in the first two columns in bold type, the end figure having been produced by rounding the sums just obtained to one figure. By differencing these figures the bold figures shown in the differences are obtained. The smoothness of the fourth differences, which normally is the check on the interpolates, is assured at this point. An unusually large difference should always stand opposite a value in which the rounding off has been in the opposite direction to the rounding off of its two immediate neighbours, as is the case with the $+5$ opposite 1·26.

* See footnote, p. 515.

TABLE F.—*Final Stage of the Interpolation.*

x	$\log x=f$	Δ'	Δ''	Δ'''	Δ^{iv}	True— Inter- polated	x	$\log x=f$	Δ'	Δ''	Δ'''	Δ^{iv}	True— Inter- polated
		+	—	+					+	—	+		
1.00	·0000000					[·0]	1.20	·0791812					
		43214							36041	302		2	[+·5]
·01	·0043214		426			—3	·21	·0827853		296		6	—2 +·7
		42788		8					35745		4		
·02	·0086002		418		+1	—3	·22	·0863598		292		+1	+·3
		42307		9					35453		5		
·03	·0128372		409		—1	+2	·23	·0899051		287		—1	+·1
		41961		8					35166		4		
·04	·0170333		401		—1	+4	·24	·0934217		283		+2	—·1
		41560		7					34883		6		
·05	·0211893		394		0	·0	·25	·0969100		277		—4	+·1
		41166		7					34606		2		
·06	·0253059		387		+1	—3	·26	·1003706		275		+5	—·5
		40779		8					34331		7		
·07	·0293838		379		—2	—2	·27	·1038037		268		—4	+·2
		40400		6					34063		3		
·08	·0334238		373		+2	—4	·28	·1072100		265		0	—·3
		40027		8					33798		3		
·09	·0374265		365		—2	·0	·29	·1105898		262		+2	—·9
		39662		6					33536		5		
·10	·0413927		359		0	[—·1]	·30	·1139434		257		0	[—·5]
		39303		6					33279		5		
·11	·0453230		353		+1	—2	·31	·1172713		252		—3	·0
		38950		7					33027		2		
·12	·0492180		346		—1	+2	·32	·1205740		250		+2	—·7
		38604		6					32777		4		
·13	·0530784		340		0	+4	·33	·1238517		246		+1	—·6
		38264		6					32531		5		
·14	·0569048		334		0	+5	·34	·1271048		241		—3	·0
		37930		6					32290		2		
·15	·0606978		328		—2	+4	·35	·1303338		239		+3	—·3
		37602		4					32051		5		
·16	·0644580		324		+4	—1	·36	·1335389		234		—3	+·1
		37278		8					31817		2		
·17	·0681858		316		—5	+6	·37	·1367206		232		+2	—·3
		36962		3					31585		4		
·18	·0718820		313		+4	+1	·38	·1398791		228		—1	—·1
		36649		7					31357		3		
·19	·0755469		306		—3	+6	·39	·1430148		225			·0
		36343		4					31132				
1.20	·0791812		302		+2	[+·5]	1.40	·1461280					[+·4]

The whole of the second difference can now be inferred and written in. The value opposite each pivotal value should be carefully compared with $\frac{1}{100} (\Delta'' - \frac{1}{12} \Delta^{iv})$, the values of Δ'' and Δ^{iv} being those of the original values. The discordance should never exceed ± 1 . Any suspicious discordance, if not satisfactorily accounted for by the direction and magnitude of the rounding off of the two neighbouring values, should be compared with the discordances at the preceding and following pivotal values, as two consecutive discordances would suggest that a wrong argument had been used in one of the tables.

At present only the end figures of the first differences are known. The full value of the leading first difference is easily found by computing

$$\cdot 1\Delta' - \cdot 0225(\Delta_0'' + \Delta_1'') + \cdot 006\Delta''' + \cdot 004(\Delta_0^{iv} + \Delta_1^{iv})$$

and adjusting the end figure to agree with its already known value. The first difference column is now completed by continuous addition of the second differences, preferably on a machine, and then the function column is produced in the same manner. A complete check on these summations is afforded by the exact reproduction of the pivotal values, the only source of error not eliminated being that of an error in transcription from the machine.

In the last column of Table F a comparison, in units of the last decimal, is made between the known values of $\log x$ and the interpolated values. The largest errors are to be expected in the neighbourhood of points where the pivotal values are in error by nearly half a unit, and it is seen that this is so, the greatest error being $\cdot 9$ units at $1\cdot 29$. If the original values are correct to within half a unit the maximum error that can occur is $1\cdot 2$ units, although it is unlikely that errors greater than $1\cdot 0$ units would occur. The error may, however, be kept definitely under 1 unit in a p -figure table if the original values are known to $p + 1$ figures, the $p + 1$ th figure being correct to within 1 or even 2 units. Three figures are then used in forming the contribution of $f_0 + n\Delta'$, the last three figures on the machine being rounded to two before being written down. The arguments for the tables are then $\frac{1}{10}(\Delta_0'' + \Delta_1'')$, $\frac{1}{10}\Delta_1'''$, etc.

Bessel's formula has been chosen partly because of the symmetrical nature of its coefficients and particularly because its third difference coefficients are very small. A Table III based on Gauss's formula, for instance, would not reach a period till 20000, and would require an interval as small as the present Table II, because its maximum coefficient is nearly eight times the maximum coefficient of the third difference in Bessel's formula. Further, the present Table III, although having the very simple period of 10000, need not be carried to that point, because the entries required for negative arguments are merely those for positive arguments, but in the reverse order. These advantages easily outweigh the extra labour of forming the double differences $\Delta_0'' + \Delta_1''$, etc. The first tables made were for Everett's formula, which ranks with Bessel's as one of the best of the central difference formulæ, but these were abandoned in favour of the simpler tables now in use.

Application of Mechanical Calculators to the Method.—It is possible, by making use of the highly-developed calculating machines now available, to eliminate practically all the mental work and writing in the method as just described. The first stage, that of writing and adding nine columns of two figures each, can be done very effectively on a tabulating machine. A number is conveyed to this type of machine by means of a card measuring $3\frac{1}{4} \times 7\frac{3}{8}$ inches, and containing 45 vertical columns, in each of which a hole may be punched in any one

of ten positions, thereby denoting one of the digits 0 to 9. When a series of cards is passed through the machine the numbers denoted by the holes punched are conveyed to counting and printing mechanism so that a printed record of the numbers used and of their total is secured.

The two principal tabulating machines are the Hollerith and the Powers. They use similar cards, but whereas the Powers machine "senses" the figures punched by a fixed mechanical device, the Hollerith "senses" by the completion of electrical circuits through the holes. It has a switch-board, not unlike that of a telephone exchange, so that the machine can be "plugged" in a few minutes to give any desired relationship between the scheme of holes punched on the card and the arrangement of the counting and printing mechanism. The great flexibility which it thus acquires renders it superior to its rival for work such as the present.

The cards are punched to represent the tables, one card being used for each value of the argument in any particular table. As the same argument may occur several times in a long run of work it is necessary to have several copies of each card. These cards, once made, are permanent, are regarded as "stock," and can be used over and over again. They are arranged in boxes, with guides separating each group with the same argument. The process of writing the figures extracted from the tables is replaced by the selection of a card for each particular argument.

The card is printed on the front in black from columns 23 to 45 only, column 23 being devoted to the table number, columns 24 to 27 to the (positive) argument, and the remaining 18 columns to the nine 2-figure entries of the tables. On the back the card is printed in red from columns 23 to 45 for negative arguments, the tabular entries for which are the complements of those on the front of the card. The right-hand side only of the card is "plugged," the card being passed into the machine black side first if the argument is positive, and red side first for a negative argument. It will be seen that column 23, which is exactly in the middle of the card, is utilised in either case for a feature that is common to both sides, namely, the table number. This has, in effect, increased the card capacity to 46 columns; otherwise it would have been necessary to make separate cards for positive and negative arguments.

The contributions of f_0 and Δ' may, if desired, be reduced to table form, but it is considered preferable to punch cards for this line as it is developed on a calculating machine in the manner already described. Each card can be used twice (back and front) since only half is used each time, so that the cost of these cards per 1000 interpolates is only 4d. It is possible to operate the calculating machine with the left hand, particularly in the case of an electric machine, and punch with the right. A good operator can punch three, four, or even five figures per second without taking her eyes from the copy.

In general, two cards are used in respect of each order of differences, one representing the leading figure or figures of the argument, the other

representing the remaining figures. The following table shows a suitable stock :

Table	Arguments	Varieties	Average No. of each Variety
II	{ 100 . . . (100) . . . 3900	39	40
	{ 1 . . . (1) . . . 99	99	20
III	{ 1000 . . . (1000) . . . 9000	9	100
	{ 10 . . . (10) . . . 990	99	20
IV	{ 1000 . . . (1000) . . . 9000	9	100
	{ 10 . . . (10) . . . 990	99	20
		Total	354

The 4000 possible arguments of Table II are therefore covered by 138 varieties of cards. When a set of these 354 "master" cards has been made any number of cards may be produced by means of a duplicating punch, which copies cards already punched at the rate of ten per minute. The above stock of about 10000 cards will permit the selection of cards representing the arguments for 1000 pivotal values (*i.e.* 10000 interpolates) without serious likelihood of any particular variety being exhausted. But if this should happen it is only necessary to use two cards whose sum is equal to the required argument. In Table II, however, one of the two cards chosen should be a multiple of 200 or of 40, as these particular cards have no rounding-off error; for the same reason one of two Table III cards should be a multiple of 100.

When one of the arguments is negative the box containing its cards is turned round. If the guide cards are appropriately labelled on the back (in red) the selection proceeds without regard to sign (unless there is a change of sign) once the boxes are arranged.

The work of the tabulating machine, through which the selected cards are passed in order, is illustrated in Plate 6, which shows the preliminary stage before the formation of Table F. A blank card is placed between each group of seven cards, while a figure 0 is punched in column 23 of each card in addition to the table number. This enables a "control" to be made by the machine on column 23, with the result that after the listing of the seventh or other card just before the blank card the machine automatically prints the totals of the nine groups of figures contained on the seven cards (but not of the table numbers or arguments), clears itself, moves the paper up four lines, and begins on the next group. The specimen illustrated, containing 650 figures and 243 additions of 2-figure numbers for the 40 interpolates, was done by the machine in 40 seconds, without any attention on the part of the operator! If totals only are to be printed, without the listing of individual cards, a switch is turned and the time required is only 30 seconds. The printing of any column of figures may be suppressed at will; this feature is utilised to prevent the printing of the "carry over" into the columns of hundreds. When the final stage is done by one of the mechanical methods described below, the space between

TABLE	ARGUMENT	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.0	27	54	81	8	35	62	89	16	43
2	3600	10	40	90	60	50	60	90	40	10
3	90	20	36	47	54	56	54	47	36	20
3	7000	20	60	90	80	00	20	10	40	80
3	510	31	41	36	20	00	80	64	59	69
4	4000	43	12	13	52	31	52	13	12	43
4	350	86	75	66	61	59	61	66	75	86
		37	18	23	35	31	89	79	78	51
1	1.1	55	40	25	10	95	80	65	50	35
2	2500	18	20	08	80	38	80	8	20	18
2	5	1	02	03	3	03	03	3	02	01
3	5000			50				50	00	00
3	720	43	58	50	29	00	71	50	42	57
4	3000	82	84	10	64	48	64	10	84	82
4	20	99	99	98	98	98	98	98	99	99
		98	03	44	84	82	96	84	97	92
1	1.2	42	64	86	8	30	52	74	96	18
2	39	9	16	20	23	24	23	20	16	09
3	4000	40	20	80	60	00	40	20	80	60
3	450	27	36	32	18	00	82	68	64	73
4	2000	22	56	07	76	66	76	7	56	22
4	150	94	89	85	83	82	83	85	89	94
		34	81	10	68	02	56	74	01	76
		2	3	8	7	3	0	7	3	0
		1	5	3	6	6	6	8	8	6
		302	296	292	283	277	275	268	265	262
1	1.3	86	32	78	24	70	16	62	08	54
2	3900	78	60	48	40	38	40	48	60	78
2	89	20	36	47	53	56	53	47	36	20
3	3000	80	40	10	20	00	80	90	60	20
3	540	32	43	38	22	00	78	62	57	68
4	1000	61	28	03	88	83	88	3	28	61
4	580	77	58	44	35	32	35	44	58	77
		34	97	68	82	79	90	56	07	78

the groups is available for the reduction to one figure and the differencing, as illustrated for the portion 1.20 to 1.30.

After use, the cards are sorted by a sorting machine and returned to the stock. The whole of the cards required for the conversion of 1000 pivotal values into 10000 interpolates could be sorted in an hour.

At this stage the second or third differences of the interpolates are known. If the difference known is the third, the second should be formed by a single summation and written down. The development of the function from its second differences by means of the Brunsviga-Dupla calculating machine has already been described in detail by the writer.* A Burroughs Class 11 machine containing an extra register, and costing about £360, affords the additional advantage of the printing of the argument, function, first and second differences, with the handling of the second differences only. In addition to the obvious time-saving resulting from the printing there is complete security from transcription error.

It need hardly be remarked that the Brunsviga-Dupla or the Burroughs may be used for the final summations even if the preliminary part is done by hand. But the Brunsviga-Dupla may also be used to considerable advantage for the nine groups of 2-figure additions. Its setting lever capacity is 15, and the fact that it possesses two registers,† one of which can be disconnected at will, enables it to be used as a 30-figure addition machine. The peculiar property of all the co-efficients, namely, that they are symmetrical in magnitude about .5, renders it unnecessary to set more than the first five columns of each table.

The 15 setting levers are divided into five fields of three each; in fact this is already done, the fields being in white and red alternately. The left-hand lever of each field is not used. The results for .1, .2, .3, .4 and .5 are to be read in order from R_2 , while the groups in R_1 are read in the reverse order to give .5, .6, .7, .8 and .9. The working process is

Setting	Operation
f_0	Add in R_2 only
Table I	„ R_2 , subtract in R_1
„ II	„ both registers
„ III	„ R_2 , subtract in R_1
„ IV	„ both registers
f_1	„ R_1 only

The settings f_0 and f_1 consist of the end figure of the function set on the middle lever of each field. At the end of each group f_1 is on the levers, and becomes the f_0 for the following group without re-setting. Table I consists of multiples of the end two figures of Δ' , the tens and units only being retained. If Δ' is negative its complement to 100 may be used as argument, or the second operation must read "Subtract in

* *M.N.R.A.S.*, **88**, 447 (1928 March).

† The reader is referred to the paper just cited for a description of the machine and the notation used.

R_2 , add in R_1 ." The agreement of the two separate values for $n = .5$ constitutes a partial check, especially on the connections and disconnections of R_2 . To add in R_2 and subtract in R_1 means a forward turn with R_2 connected, and two backward turns with R_2 disconnected. Since the process involves subtractions as well as additions the fields of three columns are not self-contained, for at times a 1 will be "stolen" from the adjacent field on the left. This effect is eliminated if, in reading the results, each field is regarded (for rounding-off purposes only) as a decimal of the field on its left.

When the original values are given to $p + 1$ figures and a p -figure table is being made, f_0 and f_1 are the end two figures of the function, while Table I consists of one-tenth of multiples of the end three figures of Δ' .

It will be seen that the Brunsviga-Dupla, which costs 110 guineas, is capable of affording all the mechanical aid required in each stage of the interpolation. The hiring of a Hollerith equipment, which costs £600 a year, would not be feasible for small undertakings, but this difficulty could perhaps be overcome by arranging for the company controlling the machine* to do the tabulations and sortings when required, for which the charge would be quite reasonable.

Accuracy of the Method.—It is possible to detect almost every type of error that can occur in working this method of interpolation. In Table G the argument for $\Delta_0'' + \Delta_1''$ was taken as 93 instead of 39. The presence of this error is revealed at once by a comparison of the end figure of the new Δ'' with its theoretical value. The error in subtraction in the column Δ'' at 1.23 would result in a failure to reproduce

TABLE G.—*Detection of Errors.*

x	Additions	f	Δ'	Δ''	Δ'''	Δ^{IV}
1.20		2		0		-4
			3		2	
.21	46	5		8		+5
			5		7	
.22	02	0		1		-3
			4		4	
.23	39	4		7		+1
			6		5	
.24	01	0		2		-1
			4		4	
.25	36	4		8		-2
			6		2	
.26	89	0		6		+6
			0		8	
.27	03	0		8		-5
			2		3	
.28	22	2		5		0
			7		3	
.29	88	9		2		+3
			5		6	
1.30		4		6		-2

* The British Tabulating Machine Company.

the original value at 1.30; an erroneous subtraction in Δ' would be detected in the same manner. The error in f at 1.26 due to incorrect rounding off (or an error in f due to incorrect addition) is detected by the fact that the large fourth difference of + 6 at this point cannot be accounted for in the normal way.

It is true that very small errors due to the use of erroneous arguments may remain undetected, but if the new differences are carefully examined by an experienced and vigilant computer, and especially if the final results are printed on a Burroughs machine, there is no danger of a serious error being passed.

A valuable check on the accuracy of the method was recently afforded by the comparison of 10,500 values of inverse trigonometrical and hyperbolic functions to seven and eight figures with the corresponding values from Hayashi's *Sieben und Mehr-stellige Tafeln*, the existence of which was not known at the time of computation. No error whatsoever was found in the work produced by this process of interpolation, although 360 (exclusive of errors of 1 in the last decimal), or nearly $3\frac{1}{2}$ per cent. of the values compared, were found in Hayashi.

Origin of the Method.—The method might appropriately be termed the end-figure method of interpolation. The idea of obtaining the end figures of second and third differences has been used on several occasions, but Mr. T. C. Hudson, then of H.M. Nautical Almanac Office, was the first to realise the advantages of obtaining the end figures of the interpolates themselves, and it is to him that the writer is indebted for the inspiration that led to the present method. Mr. Hudson obtained his end figures by graphical methods; the writer developed the 2-figure table method as being more accurate and better suited to routine work, especially as it may be done without skilled labour.

On the Occurrence of Discordant Observations and a New Method of Treating Them. By Kyrill Ogrodnikoff.

(Communicated by the Secretaries.)

1. The fact that in a large series of observations the relative number of observations differing greatly from the mean value is usually disproportionately great as compared with that predicted by a Gaussian law is very familiar to all observers. The inconvenience of the introduction of these discrepant observations into reductions is very easy to understand. Therefore a great number of different methods for avoiding this difficulty have been proposed. All of them may be fairly well divided into two classes. In the methods belonging to the first class it is generally supposed that each observer, however skilful he may be in carrying on observations, obtains in addition to "regular" results a

certain number of "erroneous" ones, and the aim of these methods is to detect the "erroneous" observations, after which they can be rejected. Among these methods the best known are those of B. Peirce, Chauvenet, and Stone. All these methods are subject to objections from theoretical as well as from practical points of view. The principal theoretical objection made by J. Bertrand is, that if we base our reductions on the Gaussian law, every value of the error, however great it may be, receives a positive probability. It is impossible therefore from theoretical reasons to take any given observation for an erroneous one. Also practical astronomers of the highest rank have made several objections to this method. The principal one is that we have a right to reject an observation only when we are aware of the imperfection of this observation at the very moment of its performance and never on the ground of *a posteriori* considerations.

In the methods of the second class no observation is rejected, but reasons are given for ascribing smaller weight to observations which differ greatly from the mean value. Among these the methods of de Morgan and Glaisher and S. Newcomb may be cited. More recently this question has been studied by R. M. Stewart. The first who introduced into calculations the hypothesis of the variability of the mean error of observations during the period of their performance is, as far as I know, S. Newcomb. The paper of Mr. Stewart is founded on the same principles as that of Newcomb. The supposition that the excess of large residuals is due to the variations of the mean error is also introduced by Newcomb. He seeks to represent the observed distribution curve of residuals by a finite sum of Gaussian curves, but the calculations to which this method leads are so very complicated that, as it seems, even Newcomb himself has never used it in practice.

In the present paper I show that if the mean error changes during the observations, the resultant frequency curve of residuals must show an excess of large residuals as compared with what is predicted by a Gaussian law, and I introduce the notion of "mean weights" of observations differing by the same amount from the mean value of the unknown quantity. For the determination of these weights I give a simple formula which permits their calculation without solving the integral equation which appears when we assume the observed frequency curve to be composed of Gaussian ones. The weights prove to be diminishing with increasing deviations. A very instructive exposition of the history of the question will be found in a fundamental book of E. Czuber, where also all necessary classical references are given.

2. It is obvious that the measure of precision of a large series of observations can only remain constant during the whole period of observations in quite exceptional cases. In general it varies from observation to observation in an unknown manner.

If we suppose that every observation belongs to some Gaussian distribution

$$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if $f(h)$ is the probability of this curve, then we shall have the following relation between $f(h)$ and the observed distribution curve $u(x)$

$$u(x) = \frac{\mu_0}{\sqrt{\pi}} \int_0^\infty f(h) h e^{-h^2 x^2} dh \quad . \quad . \quad . \quad (2)$$

μ_0 denoting a constant equal to the total number of observations, x the deviation of the observed quantity from the true value, and $\pi = 3.14$. If we divide both sides of the equation (2) by μ_0 and introduce $u(x)$ instead of $u(x)/\mu_0$, we may put in (2) $\mu_0 = 1$.

If we have two observations x_1 and x_2 belonging to two different Gaussian distributions (1), corresponding to the values h_1 and h_2 of the parameter respectively, then by combining them we shall ascribe the weight $p_1 = h_1^2$ to the first and $p_2 = h_2^2$ to the second one. The same is true in the case of any number of observations.

If we have a distribution of observations we cannot certainly say to what Gaussian curve any given one belongs, but we can try to determine the mean weight $p(x)$ of all observations differing equally from the true value of the unknown quantity.

This weight is obviously defined by the formula

$$\bar{p}(x) = \frac{\frac{1}{\sqrt{\pi}} \int_0^\infty f(h) h^3 e^{-h^2 x^2} dh}{\frac{1}{\sqrt{\pi}} \int_0^\infty f(h) h e^{-h^2 x^2} dh} \quad . \quad . \quad . \quad (3)$$

Now, taking the logarithmic derivative from both sides of (2), we find at once

$$\bar{p}(x) = - \frac{1}{2x} \frac{u'(x)}{u(x)} \quad . \quad . \quad . \quad (4)$$

$u'(x)$ denoting the derivative of $u(x)$ with respect to x , which gives us a simple means for determining $\bar{p}(x)$ from $u(x)$ without solving the integral equation (2).

From the above formulæ we can make some inferences as to the character of $u(x)$ which would allow the representation of it by the integral expression (2). From (2) we see that $u(x)$ must be symmetrical with respect to x , *i.e.* depend on x^2 only. Further, the sign of the derivative of the n th order with respect to x^2 must be the same as that of $(-1)^n$. In particular, the first derivative of $u(x)$ must be negative for positive x and vice versa, *i.e.* $u(x)$ must steadily fall off in both directions from the origin. Therefore from (4) we infer that $\bar{p}(x)$ is positive for all values of x .

It is interesting to note that when $u(x)$ is itself a Gaussian curve, then the formula (4) gives

$$\bar{p}(x) = \text{const.} = h^2;$$

i.e. we obtain an innovation only when $u(x)$ differs appreciably from the Gaussian curve.

3. For an analytical representation of the frequency functions of

observational errors it is very convenient to use the method developed by Professor C. V. L. Charlier.* According to it, every function of such a nature can be represented in the form :

$$u(x) = \mu_0(\phi(x) + \beta_3\sigma^3\phi^{\text{III}}(x) + \beta_4\sigma^4\phi^{\text{IV}}(x) + \dots) \quad (5)$$

where $\phi(x)$ is a Gaussian function

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

into which the "mean square error" σ , instead of the parameter h is introduced, $\phi^{\text{III}}(x)$ and $\phi^{\text{IV}}(x)$ are the 3rd and 4th derivatives and $\mu_0, \beta_3, \beta_4 \dots$ are constants to be determined for every particular case. As Professor Charlier states, in practice it is quite sufficient to determine the terms with the 3rd and 4th derivatives only. Therefore, the last two terms of the expansion (5) represent the deviation of the observed distribution $u(x)$ from the Gaussian one. The term with $\phi^{\text{III}}(x)$ characterises the asymmetry of $u(x)$. Since $u(x)$ is symmetrical we must put in (5) $\beta_3 = 0$, *i.e.*

$$u(x) = \mu_0(\phi(x) + \beta_4\sigma^4\phi^{\text{IV}}(x)) \quad (6)$$

It follows from the form of $\phi(x)$ that the function

$$\phi_0 = \sigma\phi(x)$$

depends upon the variable x/σ only. The same can be said with respect to the function

$$\phi_4 = \sigma^5\phi^{\text{IV}}(x).$$

Therefore $u(x)$ assumes the form

$$u(x) = \mu_0(\phi_0 + \beta_4\phi_4)/\sigma \quad (7)$$

where ϕ_0 and ϕ_4 can be tabulated according to x/σ .†

The values of the parameters μ_0 and β_4 can be computed from the values of the Pearson moments of the curve $u(x)$. If μ_s denotes the s th moment

$$\mu_s = \int_{-\infty}^{+\infty} x^s u(x) dx,$$

then μ_0 and β_4 are given by the formulæ :

$$\mu_0 = \int_{-\infty}^{+\infty} u(x) dx \quad (8)$$

and

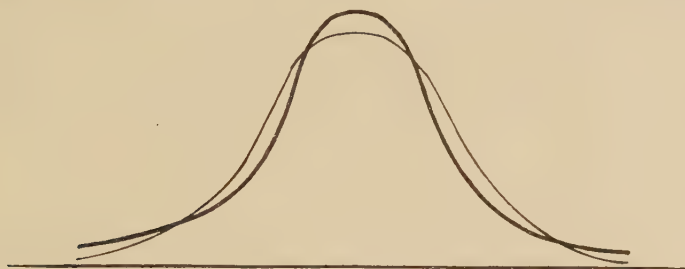
$$\beta_4 = \frac{1}{24}(\mu_4/\mu_0\sigma^4 - 3) \quad (9)$$

Two cases are to be distinguished with respect to the sign of β_4 . Professor Charlier states that the curve has a positive excess when $\beta_4 > 0$, and a negative excess in the opposite case. When $\beta_4 = 0$, $u(x)$

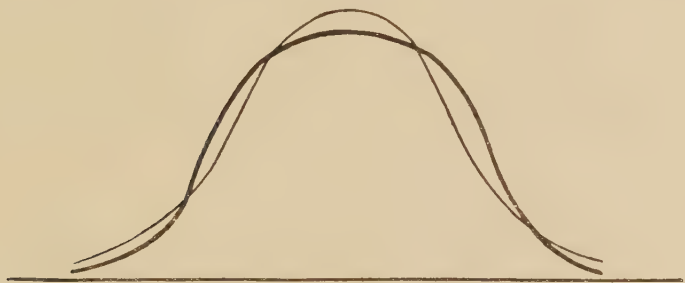
* *Lund Meddelanden*, Nris. 25, 26 och 27.

† Professor Charlier gives the tables for ϕ_0, ϕ_3 , and ϕ_4 (*loc. cit.*). In a paper on this subject, which will appear in part I, vol. v., of the *Russian Astronomical Journal* (in English), *in extenso*, I give a reproduction of these tables.

is a Gaussian curve. On the figure are represented the curves for $\beta_4 > 0$ and $\beta_4 < 0$ respectively. On both drawings the thin lines represent Gaussian curves and the thick ones the curves of Charlier. It is very important to remark that in all practical cases that I have as yet met, in every case the distribution curve had a tendency to show the positive excess, and in no case could a negative excess be detected. In the cases when there were an unusually large number of discordant observations it was invariably followed by an excess of observations near the centre



Effect of the positive excess.



Effect of the negative excess.

of the distribution, *i.e.* there always was a positive excess in its pure form.

With respect to the sign of β_4 the following theorem can be proved :—

THEOREM: If $u(x)$ be symmetrical and composed of superposed Gaussian curves, *i.e.* if the relation holds :

$$u(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(h) h e^{-h^2 x^2} dx \quad . \quad . \quad . \quad (10)$$

then $\beta_4 \geq 0$.

This may be easily verified by referring to the expression for β_4 in the formula (9). We can put without loss of generality $\mu_0 = 1$. Then the sign will be the same as the sign of the expression :

$$\int_{-\infty}^{+\infty} x^4 u(x) dx - 3 \left[\int_{-\infty}^{+\infty} x^2 u(x) dx \right]^2.$$

Substituting the expression for $u(x)$ from the formula (10), performing the integration with respect to x , and referring to the conditions which the function $f(h)$ fulfils, which are

$$0 \leq f(h) \leq 1 \quad \text{and} \quad \int_0^\infty f(h) dh = 1,$$

we obtain, after applying to the second term the Bessel's inequality, the desired inequality.

It is convenient, therefore, to call the coefficient β_4 the coefficient of *non-homogeneity* of the system of observations.

It can be also easily proved that the mean weights $\bar{p}(x)$, as deduced from the form (10) of the function $u(x)$, must steadily fall off toward both sides from the centre. Taking the derivative of $\bar{p}(x)$ we see that its sign is the same as the sign of the expression :

$$-2x \left\{ \int_0^\infty h^5 f(h) e^{-h^2 x^2} dh \times \int_0^\infty h f(h) e^{-h^2 x^2} dh - \left[\int_0^\infty h^3 f(h) e^{-h^2 x^2} dh \right]^2 \right\} \quad (11)$$

Applying once more Bessel's inequality to the difference in the round brackets we see that it is positive for all values of x , whence the desired theorem follows.

The limit toward which $\bar{p}(x)$ tends for x tends to $\pm \infty$ can be obtained by a reasoning which is valid for all cases when $u(x)$ has the form of a Gaussian curve multiplied by a polynomial. Taking the general case, we can suppose that the Gaussian curves from the superposition of which $u(x)$ arises are available for all values of h lying in a certain interval, the limits of which, h_1 and h_2 , can assume all positive values, provided that $h_1 < h_2$. h_2 may be equal to ∞ , h_1 to zero.

The equation determining $\bar{p}(x)$ we shall write down in the form :

$$\int_{h_1}^{h_2} h^3 f(h) e^{-h^2 x^2} dh = \bar{p}(x) \int_{h_1}^{h_2} h f(h) e^{-h^2 x^2} dh.$$

We can consider in this equation x as a parameter and h_1 and h_2 as independent variables. Denoting by $\frac{\delta}{\delta h_1}$ the derivative with respect to the variable h_1 , we have

$$-h_1^3 f(h_1) e^{-h_1^2 x^2} = -\bar{p}(x) h_1 f(h_1) e^{-h_1^2 x^2} + \frac{\delta \bar{p}(x)}{\delta h_1} \int_{h_1}^{h_2} h f(h) e^{-h^2 x^2} dh.$$

Dividing both sides by $e^{h_1^2 x^2}$ and substituting $u(x)$ in the place of the integral on the right side of this equation, we obtain :

$$h_1 f(h_1) \bar{p}(x) = h_1^3 f(h_1) - \frac{\delta \bar{p}(x)}{\delta h_1} e^{h_1^2 x^2} u(x).$$

It is obvious that if h_0 denotes the value of the parameter of the Gaussian curve entering in the expression of $u(x)$,

$$u(x) = e^{-h_0^2 x^2} P(x)$$

where $P(x)$ is a polynomial, the double inequality holds :

$$h_1 < h_0 < h_2.$$

If, therefore, x tends to $\pm \infty$, the factor $e^{h_1^2 x^2} u(x)$ must vanish. It follows then that

$$\lim_{x=\infty} \bar{p}(x) = h_1^2 = p_1,$$

i.e. to the minimal weight of all the observations. If $p_2 = 0$, then of course the limit of $\bar{p}(x)$ is also zero.

It would be very interesting to study, from the point of view of the hypothesis of elementary errors, what can be learned in the general case when in the expression of $u(x)$ we have instead of a polynomial of finite degree an infinite series. But for practical purposes the case considered is, I think, quite sufficient.

4. By the method of Professor Charlier an interpolation formula for the mean weights can be constructed. Substituting the expression (6) for $u(x)$ in (4) we obtain :

$$\bar{p}(x) = \frac{-1}{2x} \frac{\phi' + \beta_4 \sigma^4 \phi^V}{\phi + \beta_4 \sigma^4 \phi^{IV}}.$$

But, owing to the fact that each derivative of ϕ is of the form

$$\phi^{(n)} = \phi p_n,$$

where p_n is a polynomial of Hermite, we obtain after insignificant reductions :

$$\bar{p}(x) = \frac{1}{2\sigma^2} \frac{1 + \beta_4(z^4 - 10z^2 + 15)}{1 + \beta_4(z^4 - 6z^2 + 3)} = p_0 \frac{f_1(z)}{f_2(z)} \quad (12)$$

where instead of x/σ a new variable z is introduced, and p_0 denotes the weight which must be ascribed to the Gaussian curve which we should obtain by putting $\beta_4 = 0$.

It is clear that the interpolation formula (12) is valid only in a finite interval. As we have seen above, the mean weights must steadily fall off. If, therefore, by applying the interpolation formula, a limit is reached where the weights have a minimum, or become negative, then this point may be considered as an upper limit for the validity of this interpolation process.

Taking the derivative of both sides of (12) we find that until $\beta_4 < \frac{1}{15}$ the weights really decrease from the origin to a point where they have a minimum. This point is given by the formula :

$$z = \pm \sqrt{3 + \sqrt{1/\beta_4 - 6}} \quad (13)$$

5. Hereafter are given two examples to which this method was applied. The first one consists of 684 observations of the moments of contact in the passages of Mercury across the solar disc from 1677

to 1881, compiled and discussed by S. Newcomb.* The corresponding data are given in the following Table I. :—

TABLE I.

Mean Value of Residuals.	Actual Number.	Probable Number.	A-P.	Mean Weight ($f_1(z)/f_2(z)$).
0 ^s	147	137	+10	1.58
5	221	240	-19	1.57
10	129	166	-37	1.52
15	77	88	-11	1.24
20	38	36	+ 3	0.68
25	23	12	+11	0.41
27	49	5	+44	..

The first column gives the deviation of the observed moments from the theoretical ones, the second the observed number of observations, the third one the number predicted by the Gaussian law, the fourth one the differences between the corresponding numbers of the third and fourth columns, and, finally, the fifth one gives the mean weights which I determined. These observations, as Newcomb has stated, are distinguished by a disproportionate number of large residuals. We see from the fourth column that here we have a pure positive excess. After some precautions taken in order not to introduce a systematic error through observations with large residuals, for which the probability is great that they originate from mistakes in counting seconds,† I have obtained for μ_0 , σ , and β_4 the following values :—

$$\mu_0 = 644, \quad \sigma = 10^8.52, \quad \beta_4 = 0.0568.$$

From Table I. we see that for small residuals the weights remain nearly constant. By proceeding to greater residuals they diminish more and more rapidly. But at $x/\sigma = 2.53$ they have a minimum, and therefore this point may be regarded as an upper limit up to which our conclusions are valid.

As a second example I have taken the distribution of 791 radial velocities of K stars from the two large Catalogues : J. Voûte, "First Catalogue of Radial Velocities," and W. Adams and A. Joy, "The Radial Velocities of 1013 Stars."‡ This is an example of a quite different nature. Nevertheless, the introduction of variable weights into statistical investigations of stellar motions will be, I hope, as convenient as in the case of observational errors.

From the point of view of our hypothesis our stellar universe appears, not as a collection of individual stars, but as a multitude of stellar groups having different average velocities, the internal distribution in which follows, within each one, the Gaussian law. This hypothesis appears quite natural owing to the fact that the average velocities of

* Astronomical papers of the *American Ephemeris*, vol. i., 1884.

† For more details see my paper in the *Russ. Astron. Jour.* (*loc. cit.*).

‡ Mt. W. Contributions, No. 258.

stars of the different spectral types are not the same. It is justified by the fact that the distribution of stellar velocities shows a very strongly pronounced positive excess. If we have two stars s_1 and s_2 belonging to two different Gaussian distributions, it is reasonable to ascribe greater weight to that star for which h is greater. For the greater is h the more exactly is defined the centre of the corresponding Gaussian curve.* The data relating to this distribution are given in Table II. The designations of the columns are the same as in Table I.

TABLE II.

Velocity.	Actual Number.	Probable Number.	A-P.	Mean Weights.
0-5 km./sec.	74	65	+ 9	1.37
5-10 "	135	127	+ 8	1.37
10-15 "	116	119	- 3	1.36
15-20 "	98	107	- 9	1.35
20-25 "	76	92	-16	1.33
25-30 "	73	77	- 4	1.29
30-35 "	32	61	-29	1.24
35-40 "	33	46	-13	1.16
40-45 "	24	34	-10	1.06
45-50 "	14	24	-10	0.93
50-55 "	12	16	- 4	0.81
55-60 "	12	10	+ 2	0.70
60-65 "	6	6	0	0.63
65-70 "	4	4	0	0.59
70-75 "	2	2	0	0.59

For the values of the parameters μ_0 , σ , and β_4 I have obtained the following values:—

$$\mu_0 = 791, \quad \sigma = 24.39 \text{ km./sec.}, \quad \beta_4 = 0.0342.$$

A last remark which might be made is on the mode of introducing mean weights into computations according to the method of least squares. It would be, at first sight, very promising to consider them as variable functions of the quantities sought for, multiply each conditional equation by $\sqrt{\bar{p}(x)}$, and transform the equations thus obtained into linear ones by the usual method. But on several grounds it is more advisable to consider them as constants whose numerical values have been determined from the previous approximation. The principal reason in favour of this is that by transforming conditional equations we are substituting in the place of $\bar{p}(x)$ a linear approximation which can, if p_2 is small, assume *negative values*, however small the corrections to the unknown quantities. For more details see my paper in *Russ. Astron. Jour. (loc. cit.)*.

* The greater is h , the more rapidly falls off the Gaussian curve.

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*Occultations of Stars by the Moon, observed at the Royal Observatory,
Pino Torinese, during the period 1926 September-1927 December.
By Raffaele Invrea, D.Sc.*

No.	Date.	Star.	Mag.	Phase.	G.M.T.			Tele- scope.	Ob- server.
					h	m	s		
1	1926 Sept. 21	33 Piscium	4.7	D	19	8	51.6	M	I
2	21	33 Piscium	4.7	R	20	11	38	M	I
3	Dec. 19	μ Geminorum	3.2	D	18	26	15.6	Z	I
4	19	μ Geminorum	3.2	R	19	27	10*	Z	I
5	1927 Feb. 9	162 B Tauri	6.3	D	20	20	0	Z	I
6	13	192 B Geminorum	6.3	D	20	4	33*	M	F
7	Apr. 16	80 Virginis	5.6	D	19	25	54.9	M	I
8	26	α Capricorni	4.8	R	3	48	5.5	D	I
9	May 5	87 B Geminorum	5.8	D	19	53	26.1	M	F
10	5	87 B Geminorum	5.8	D	19	53	26.7	D	I
11	June 7	ν Virginis	4.2	D	23	15	43.1	M	I
12	13	58 G Scorpii	6.2	D	21	43	36.9	M	I
13	13	58 G Scorpii	6.2	D	21	43	37.0	Z	V
14	July 4	W.Z.C. 747	6.8	D	20	55	17.8	M	I
15	4	W.Z.C. 747	6.8	D	20	55	18	D	V
16	13	126 B Sagittarii	5.7	D	20	43	48.6	M	I
17	13	126 B Sagittarii	5.7	D	20	43	48.7	Z	Vc
18	13	126 B Sagittarii	5.7	R	22	1	16*	M	I
19	Aug. 23	87 B Geminorum	5.8	R	3	19	54.7	Z	I

Note.—Times marked with an asterisk (*) are somewhat uncertain. No. 34 was observed during Total Lunar Eclipse of 1927 December 8, under very unfavourable weather conditions.

No.	Date.	Star.	Mag.	Phase.	G.M.T.			Tele- scope.	Ob- server.
					h	m	s		
20	1927 Aug. 31	B.D. -6° 3892	7.4	D	19	29	37.6	M	I
21	31	W.Z.C. 885	7.0	D	19	43	29.3	M	I
22	Sept. 9	154 B Capricorni	6.1	D	19	47	53.5	M	I
23	10	τ Aquarii	4.1	D	19	50	8.5	M	I
24	10	τ Aquarii	4.1	R	20	57	51*	M	I
25	Oct. 3	Cordoba A. 12810	7.4	D	17	53	28.0	M	I
26	3	λ Sagittarii	2.9	R	18	20	9.5	M	I
27	11	39 B Arietis	6.5	R	19	11	9.5	M	V
28	11	64 Ceti	5.8	D	21	36	25.3	M	V
29	11	ξ^1 Ceti	4.5	D	22	34	25.4	M	V
30	30	Cordoba A. 12580	7.7	D	17	54	42.2	M	I
31	30	Cordoba A. 12592	8.5	D	18	2	58.4	M	I
32	Nov. 2	χ Capricorni	5.3	D	21	49	50.5	M	F
33	3	161 B Capricorni	6.4	D	20	25	47.7	M	F
34	Dec. 8	B.D. $+21^{\circ}$ 761	9.5	D	17	46	18*	M	I
35	30	33 Piscium	4.7	D	16	50	7.4	M	I
36	30	33 Piscium	4.7	R	18	2	48	M	I

Instruments used :—

M = 30-cm. Merz-Cavignato refractor.

Z = 15-cm. Zeiss refractor.

D = 10-cm. Dollond refractor.

Observers :—

V = L. Volta.

Vc = P. Vocca.

I = R. Invrea.

F = M. Ferrero.

Pino Torinese :

1928 March 10.

Occultations observed at Herne in 1927. By B. M. Peek.

Co-ordinates of Observatory. $\left\{ \begin{array}{l} \text{Longitude } 0^{\text{h}} 4^{\text{m}} 27^{\text{s}}.7 \text{ E.} \\ \text{Latitude } 51^{\circ} 21' 18'' \text{ N.} \end{array} \right.$

The occultations tabulated below were observed at Herne in 1927. The position of the observatory is given for the refractor, but the reflector is only about 30 feet away to the S.S.E. As in former years, the time as recorded by the stop-watch has in every case been corrected by 0^s.5 to allow for the lag in starting; in certain cases, when it has been known that there was more lag than usual, further small corrections have been applied.

Date 1927.	Star.	Mag.	Phase.	G.M.T.			Instrument and Power.	Remarks.
				h	m	s		
Feb. 9	162 B Tauri	6.3	D	20	24	2.5	R 135	Glided.
9	W.Z.C. 247	6.8	D	22	28	24.1	R 135	
12	W.Z.C. 470	6.7	D	18	7	41.1	R 135	
13	192 B Geminorum	6.3	D	19	58	2.0	S 200	
Mar. 8	B.D. +16° 514	7.9	D	22	15	2.7	R 135	
10	Lalande 10607	6.0	D	21	30	13.3	R 135	
12	B.D. +23° 1744	6.4	D	22	10	49.5	R 96	
15	107 B Leonis	6.3	D	21	12	5.0	R 190	
May 6	W.Z.C. 536	7.1	D	21	41	46.3	R 135	
7	W.Z.C. 603	6.9	D	20	52	43.2	R 135	
June 3	B.D. +21° 1844	6.3	D	22	7	40.3	R 135	{ Star gave a preliminary wink and then glided out.
10	598 B Virginis	6.1	D	22	15	13.4	R 190	
12	34 Libræ	6.0	D	23	7	14.3	R 135	{ "Comes" a few seconds earlier as contact oblique.
13	58 G Scorpii	6.2	D	21	33	53.2	R 135	
July 4	W.Z.C. 747	6.8	D	20	45	3.8	R 135	{ Aug. 4. Clock error uncertain by 0.2. D. seemed gradual; at any rate not normal.
Aug. 4	95 Virginis	5.4	D	20	35	21.9	R 135	
9	70 B Sagittarii	6.4	D	21	3	29.8	R 135	
Oct. 3	λ Sagittarii	2.9	R	18	6	4.4	R 135	{ Good observation.
Dec. 5	39 B Arietis	6.5	D	16	16	15.0	S 200	
5	64 Ceti	5.8	D	19	34	25.7	S 200	
5	ξ ¹ Ceti	4.5	D	20	36	29.1	S 200	

Instruments. { R = 150-mm. Zeiss equatorial refractor.
 { S = 12 $\frac{1}{4}$ -in. equatorial reflector.

Erratum.

P. 487, footnote. The R.A. of the radiant of meteors from Skjellerup's comet should be 33 degrees, not 122 degrees. The mistake arose through wrong copying of a sign.
 A. C. D. C.